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DETERMINATION OF THE MOTION OF FREE NONLINEAR SINGLE DEGREE OF FREEDOM VIBRATION SYSTEMS

A THESIS

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Russell Reid Kearl

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SUMMARY

In the study of vibrating systems one often encounters non-linearities in restoring and damping forces and it becomes desirable to predict the behavior of such systems. Damping forces, which can be non-negative functions of displacement and velocity of both the "coulomb" and "turbulence" type are studied to determine their effect upon free single degree of freedom vibration systems with various types of nonlinear restoring forces. The equations of motion are written and integrated once to obtain equations which describe the systems motion in the phase plane. These phase plane equations are then solved by graphical and/or analytical methods to find successive amplitudes of vibration. In the case of a "coulomb" damped system the graphical solution for consecutive amplitudes involves the superposition of two curves by an overlay technique. For "turbulence" damped systems the graphical solution involves superimposing a system of parallel lines on a set of asymetric curves.

When the phase plane equations were solved analytically for successive amplitudes the resulting equations are given, and in the remaining cases the results appear in the form of curves which can be solved by the graphical methods outlined in the text.

CHAPTER I

INTRODUCTION

The subject of vibrations is concerned with the analysis of the behavior of systems acted upon by oscillatory forces. In the world of engineering, such oscillatory forces are frequently encountered. It is desirable to understand and be able to predict the motion of such systems.

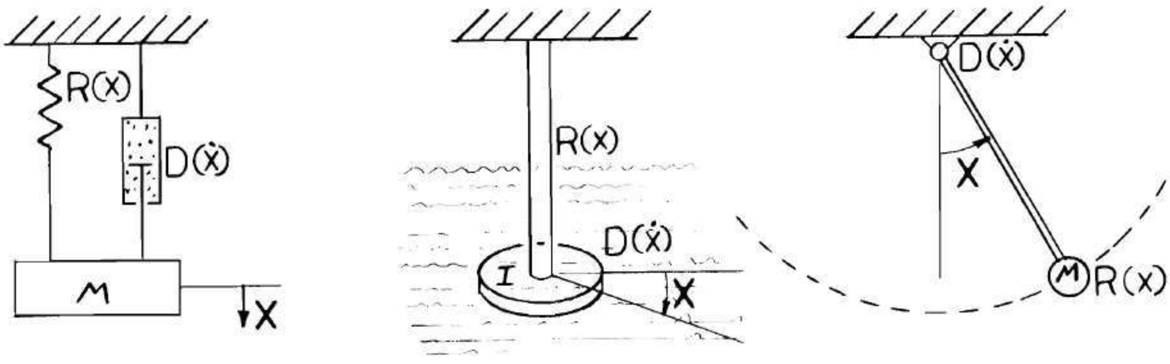

Figure 1.1 Typical Single Degree of Freedom Systems.

This thesis is concerned with the analysis of the systems such as those shown in Figure 1.1. These systems have been analyzed and studied extensively in many undergraduate vibration textbooks for the special case where the spring and damping forces are linear functions of displacement and velocity respectively. The case in which the damping force is proportional to the velocity is commonly referred to as "viscous" damping and is present in many fluid dampers where fluid velocities are small. This type of damping also appears in the motion of conductors in a magnetic field. We wish to study the behavior of a system in which the restoring force and/or the damping force are nonlinear.

Damping forces of the three types listed below will be considered in the work that follows:

1. "Coulomb" type, i.e., dry friction

2. "Turbulence" type, i.e., proportional to the square of the velocity, and

3. A combination of "coulomb" and "turbulence" types.

In all cases the results for a linear restoring force will be developed first and will be used to compare the behavior of the systems with nonlinear restoring forces. In the analysis that follows we consider the action of the elements in the system to be "lumped together." This means that a spring has elastic properties but is considered to be massless and free of internal damping, and that dampers have no mass or elastic contributions.

The vibrators in Figure 1.1 are said to be of "single degree of freedom" because at any given time a single generalized coordinate will describe the configuration of the system completely. It should be recognized that although many systems may have more than one degree of freedom they can often be studied as though they were simple systems with one degree of freedom, and in spite of the simplification, acceptable results can be obtained. In a system with two or three degrees of freedom it may be found that motion in one degree predominates. In such a case the system behavior may well be approximated by a system with only one degree of freedom. Surprisingly enough a large number of complicated systems can be treated in this manner with good engineering results.

If the vibrator is displaced from its position of equilibrium by a distance $x = 0$ it will be acted upon by a spring which will tend to return the system to its position of equilibrium. Let us assume the restoring force of the spring can be expressed as a function of the displacement only, i.e., $F_s = r(x)$. The statement that the system is "restored" to its position of equilibrium means (if $x = x_o = 0$ is the equilibrium position) that the product of the displacement and the restoring force will be non-negative.

We will assume that the damping force can be expressed as a function of displacement and velocity, i.e., $F_d = d(x,\dot{x})$*. We will also assume that the direction of the damping force is such that it opposes the motion, i.e., the product $\dot{x}d(x,\dot{x})$ must be non-negative.

Applying Newton's Second Law to the systems in Figure 1.1 we find that the equation of motion becomes

$$m\ddot{x} + d(x,\dot{x}) + r(x) = 0 \quad . \tag{1.1}$$

Let us write the expression $r(x)$ as $r(x) = k^r(x)$ where k is the spring constant. For restoring forces of a polynomial type the constant k is the slope of the curve $r(x)$ at $x = 0$. We will also assume that the damping force can be written in the form $d(x,\dot{x}) = cf(x)\, D(\dot{x})$ where c is a constant and $f(x)$ is a non-negative function. Using the

---

*The notation $\dot{x} = \frac{dx}{dt}$ will be used throughout the text.

above notation equation (1.1) can be written

$$m\ddot{x} + cf(x)\, D(\dot{x}) + k\, R(x) = 0 \quad . \tag{1.2}$$

Thus, the equation of motion for a system with a linear spring and "viscous" damping would read

$$m\ddot{x} + cf(x)\, \dot{x} + k\, x = 0 \quad . \tag{1.3}$$

In the investigation of motion of the systems in Figure 1.1 it will be convenient to find an expression for the velocity as a function of the displacement x. We define the velocity $\dot{x}$ as the variable y. The equation of motion (1.2) is a second order differential equation which we will reduce to two first order equations as follows. Since

$$\frac{d^2x}{dt^2} = \frac{d\dot{x}}{dt} = \frac{dy}{dt} = \frac{dy}{dx}\,\dot{x} = \frac{dy}{dx}\,y$$

we can now rewrite equation (1.2) as the two equations $\dot{x} = y$ and

$$\frac{dy}{dx} = -\left(\frac{c\; f(x)\; D(y) \;+\; k\; R(x)}{my}\right). \tag{1.4}$$

The (x,y) plane is called the phase plane. Note that the motion of the vibrator can be described by a curve in the (x,y) plane. The phase plane shows both in magnitude and direction the velocity of the system for any given displacement. Periodic motion with no damping would be represented by a closed curve or phase trajectory in the phase plane. If, however, damping is present in a periodic system, the curve would perhaps become a spiral converging on the equilibrium position $(x_o, 0)^*$.

Equation (1.4) gives the slope of the phase trajectory at the point $P(x,y)$ and will be called the differential equation of the phase plane. When the slope at $P(x,y)$ is given value, P, the point $P(x,y)$ is said to be an ordinary point, but if the expression for the slope becomes indeterminate, i.e., both numerator and denominator become zero, the point $P(x,y)$ is called a "singular point." These so-called singular points correspond to points of equilibrium in the mechanical system. Investigation of the motion around these singularities gives considerable insight into the behavior of the system.

In the chapters that follow we will find expressions for $y(x)$ and will indicate how these expressions can be used to find successive amplitudes of vibration.

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*The equilibrium position will be defined as the position of the vibrator in its undeflected state if no damping is present. We note, however, that coulomb damped systems may be in "equilibrium" at positions other than that where the spring exerts no force, since the friction force will always equal the spring force until the limiting value is reached and motion begins.

CHAPTER II

FREE OSCILLATIONS WITH COULOMB DAMPING

Consider the damping function introduced in equation (1.2) to be of the form $cf(x)\ D(\dot{x}) = c_o\ f(x)\ \mathrm{sgn}\ \dot{x}$, where

$$\mathrm{sgn}\ (\dot{x}) = \begin{Bmatrix} +1 \text{ for } \dot{x} > 0 \\ -1 \text{ for } \dot{x} < 0 \end{Bmatrix} .$$

Negative functions $f(x)$ are not permitted in this analysis since this would correspond to negative damping or energy input which could cause self-excited oscillations. The above type of damping could arise in a physical system of the form shown in Figure 2.1. The mass in this system is subjected to a normal force which is a non-negative function of x.

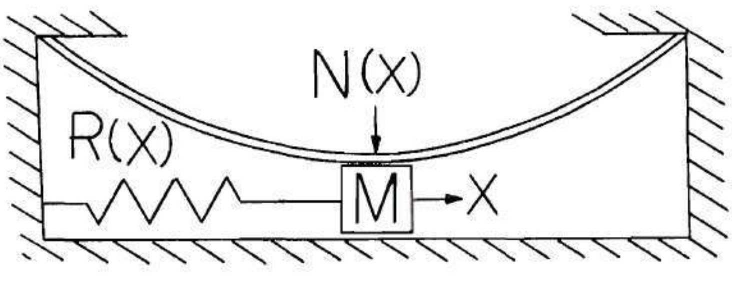

Figure 2.1 System with Coulomb Damping Dependent Upon Displacement

Since the equation of motion is a force balance, and $\mathrm{sgn}(\dot{x})$ is dimensionless, the term $c_o\ f(x)$ has the dimension of a force. For example, if the mass were sliding on a surface with a constant coefficient of kinetic friction μ_k and were subjected to a normal force $N(x)$, the term $c_o\ f(x)$ would be $\mu_k\ N(x)$. With coulomb damping defined as above, the equation of motion (1.2) becomes

$$m\ddot{x} + c_o\ f(x)\ \mathrm{sgn}\ \dot{x} + k\ R(x) = 0 \quad . \tag{2.1}$$

The differential equation of the phase plane (1.4) becomes

$$\frac{dy}{dx} = -\frac{c_o\ f(x)\ \mathrm{sgn}\ y + k\ R(x)}{my} \quad . \tag{2.2}$$

For convenience, let us introduce the notation $\frac{k}{m} = \omega_o^{\ 2}$. In the case of a single degree of freedom system without damping and a linear restoring force, ω_o is the natural circular frequency of vibration. For nonlinear systems, ω_o is a constant and may have no physical significance. We also introduce the notation $\frac{c_o}{m} = \lambda\ \omega_o^{\ 2}$ where λ is a constant. From the above notation we see that $\lambda = \frac{c_o}{k}$ and that it has the dimensions of displacement. We call λ the "decrement of the vibration" and will find that it takes on a special meaning in the case where $f(x) = 1$, and $R(x) = x$. In the case of a vibrator with constant coulomb damping and linear restoring force the difference in amplitude between any two consecutive amplitudes will be found to be 4λ which is a well known result.

We can now rewrite equation (2.2) as

$$\frac{dy}{dx} = \frac{-\omega_o^{\ 2}\ [\lambda\ f(x)\ \mathrm{sgn}\ y + R(x)]}{y} \quad . \tag{2.3}$$

We now assume that the vibration takes place about a "singularity" or point of equilibrium of the system, and let x_o be that singularity. Recall that the system is "restored" to its equilibrium position, i.e., $xR(x) \geq 0$, and that it is "truly damped," i.e., $\dot{x}f(x) \operatorname{sgn} \dot{x} > 0$. In terms of the physical system this means that energy is taken out through the damper and the singular point x_o is a stable point of equilibrium.

Let us now displace the system to an initial position $x_1 > x_o$ and release it from rest. The velocity will become negative (sign convention in Figure 1.1) as the mass accelerates back towards its equilibrium position. If the damping force is not too large the mass will overshoot its equilibrium position and will stop at a point $x_2 < x_o$; the velocity will then become positive as the mass returns. From the point $x = x_1$ where $y = 0$ to the point $x = x_2$ where again $y = 0$, the motion can be found from equation (2.3), which we rewrite as

$$y dy = - \omega_o^2 \, [\lambda \, f(x) \operatorname{sgn} y + R(x)] \, dx \quad . \tag{2.4}$$

Integrating between the limits $x = x_1$, $y = 0$, and $x = x$, $y = y$ where $\operatorname{sgn} y = -1$, we have

$$\int_o^y \xi \, d\xi = - \omega_o^2 \left(- \lambda \int_{x_1}^{x} f(\eta) \, d\eta + \int_{x_1}^{x} R(\eta) \, d\eta \right) \quad . \tag{2.5}$$

Thus, for motion in the first half cycle the expression for $\frac{y}{\omega_o}$ is

$$\frac{y}{\omega_o} = - \sqrt{2 \left[- \lambda \int_x^{x_1} f(\eta) \, d\eta + \int_x^{x_1} R(\eta) \, d\eta \right]} \quad . \tag{2.6}$$

The above equation holds while the mass moves from x_1 to x_2, then the velocity will change sign. During the motion from the point x_2 to the next reversal at x_3, we have from equation (2.4)

$$\int_0^y \xi \, d\xi = - \omega_o^2 \left(\lambda \int_{x_2}^{x} f(\eta) \, d\eta + \int_{x_2}^{x} R(\eta) \, d\eta \right) . \tag{2.7}$$

Thus, the motion during the second half cycle can be described by the equation

$$\frac{y}{\omega_o} = + \sqrt{2 \left[\lambda \int_x^{x_2} f(\eta) \, d\eta + \int_x^{x_2} R(\eta) \, d\eta \right]} . \tag{2.8}$$

We can find an expression for motion in the third half cycle in a similar manner as was done above to obtain the equation for motion in the first half cycle. It is

$$\frac{y}{\omega_o} = - \sqrt{2 \left[- \lambda \int_x^{x_3} f(\eta) \, d\eta + \int_x^{x_3} R(\eta) \, d\eta \right]} . \tag{2.9}$$

We can now write a general equation which holds for motion during the n^{th} half cycle (where $n = 1, 2, 3, 4, \ldots . N$) by the method outlined above for the first three half cycles. It is

$$\frac{y}{\omega_o} = (-1)^n \sqrt{2 \left[(-1)^n \lambda \int_x^{x_n} f(\eta) \, d\eta + \int_x^{x_n} R(\eta) \, d\eta \right]} . \tag{2.10}$$

CHAPTER III

CONSTANT COULOMB DAMPING

In the work that follows in this chapter the preceding theory will be specialized to the case where $f(x) = 1$ which means that the Coulomb damping is a constant. Constant Coulomb damping may be only an approximation to a more complicated damping law, but this simplification usually gives acceptable engineering results.

We begin by defining the function

$$V_o(x) = \int_{x_o}^{x} R(\eta)\, d\eta \tag{3.1}$$

so that motion during any half cycle can be described by equation (2.14) rewritten in the form

$$\frac{y}{\omega_o} = (-1)^n \sqrt{2[(-1)^n \lambda(x_n - x) + V_o(x_n) - V_o(x)]} \quad . \tag{3.2}$$

We notice that at both ends points of any half cycle the velocity is zero. Thus, at $x = x_1, x_2, x_3, \ldots x_{n+1}$ the expression under the radical in equation (3.2) is zero. We write from equation (3.2) for the first half cycle at the point $x = x_2$, $y = 0$

$$\lambda(x_1 - x_2) = V_o(x_1) - V_o(x_2) \quad . \tag{3.3}$$

Notice that equation (3.3) can be written as

$$\lambda = \frac{V_o(x_1) - V_o(x)_2}{x_1 - x_2} \quad , \qquad (3.4)$$

thus the unknown values $V_o(x_2)$ and x_2 can be found by drawing a straight line with slope of λ in the $[x, V_o(x)]$ plane through the point $[x_1, V_o(x_1)]$. The point $[x_2, V_o(x_2)]$ will be found at the intersection of this line and the $V_o(x)$ curve. This construction is shown in Figure 3.1 below.

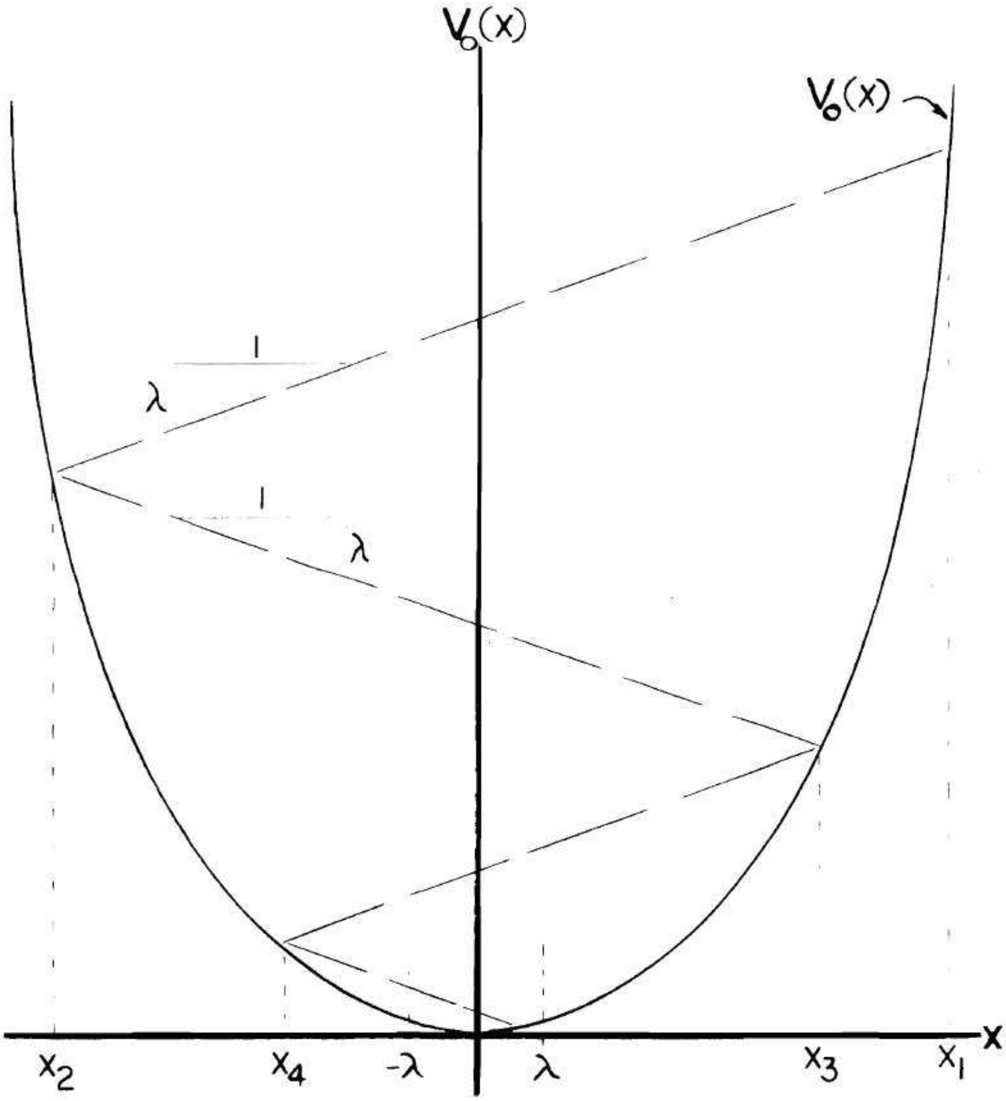

Figure 3.1 Graphical Method of Finding Successive Amplitudes of Vibration With Constant Coulomb Damping.

For the second half cycle motion follows equation (3.2) and the velocity will become zero when the mass reaches the point $x = x_3$. At the point $x = x_3$ we can write equation (3.3) as

$$- \lambda = \frac{V_o(x_2) - V_o(x_3)}{x_2 - x_3} \quad . \tag{3.5}$$

The unknown values $V_o(x_3)$ and x_3 can be found by drawing a straight line with slope $- \lambda$ through the point $[x_2, V_o(x_2)]$ and locating the intersection of this line with the curve $V_o(x)$.

Successive amplitudes can thus be found by alternately drawing straight lines of slope λ and $- \lambda$ upon the curve of $V_o(x)$. The method continues until we reach the amplitude where the absolute value of the restoring force is less than or equal to λ. When we have reached the above point motion will stop since this corresponds to a restoring force that is smaller than the damping force which would be present if the mass were to move.

An alternate method for finding successive amplitudes is described below. A straight line is drawn with a slope of λ on a sheet of transparent plastic or thin paper. This transparent sheet or overlay is then superimposed upon the graph of the function at $[x_1, V_o(x_1)]$, the next amplitude is found where the overlay crosses the $V_o(x)$ curve. Turning the overlay face down would give a line with slope $- \lambda$.

It should be remembered that although the initial displacement used to develop the graphical method was positive, the method is general and applies also for any initial amplitude which is negative. In the case of a negative initial amplitude we begin at the point

$[x_1, V_o(x_1)]$ and draw a straight line with slope $-\lambda$, and continue as before.

In the paragraphs that follow we will select specific restoring forces which might be encountered in practical applications. In these specific examples we will write the equation of motion, the differential equation of the phase plane, and the expressions for $V_o(x)$ and y/ω_o. In addition, a curve of $V_o(x)$ will be drawn in each case that can be used by the reader to determine successive amplitudes of vibration.

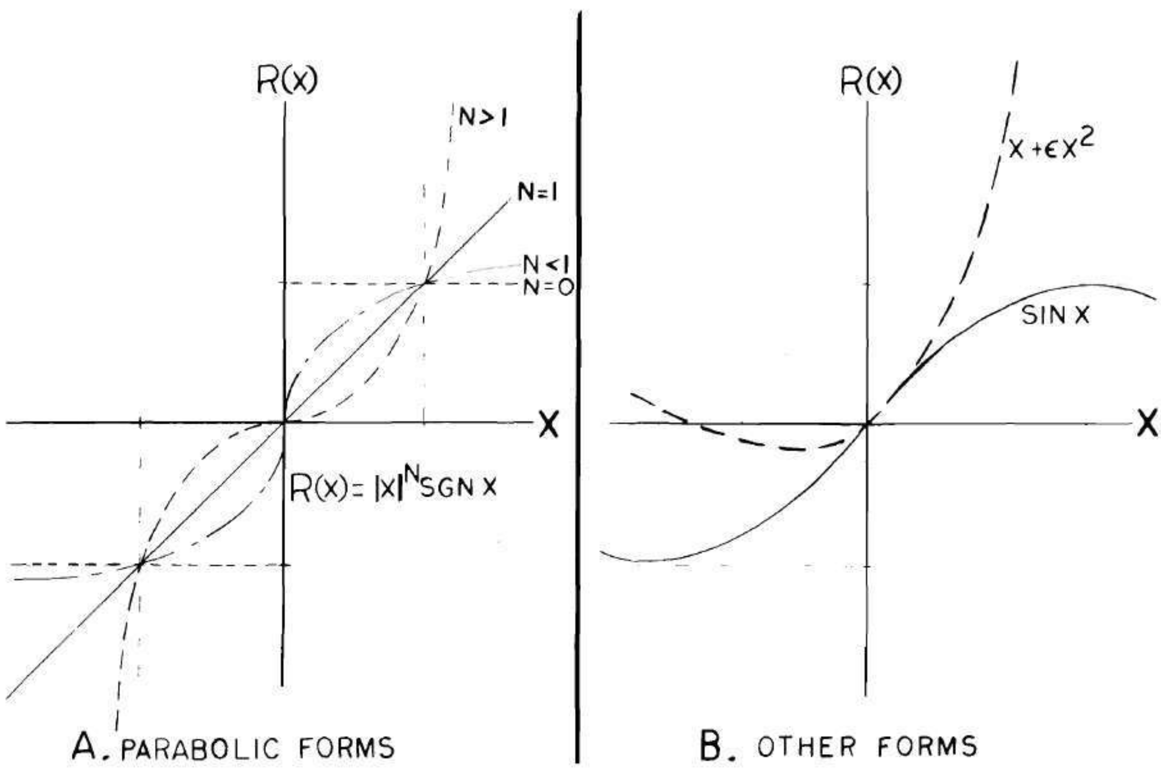

Figure 3.2 Typical Restoring Forces of Nonlinear Systems.

Figure 3.2 shows the types of restoring forces which will be discussed in their graphical form. They are as follows:

$$R(x) = x$$
$$R(x) = |x|^m \operatorname{sgn} x \quad m \ge 0$$
$$R(x) = |x|^m + \epsilon|x|^p \quad m \geqq 0,\ P \ge 0$$
$$R(x) = \sin x$$
$$R(x) = \tanh x$$
$$R(x) = x + \epsilon x^2 \text{ (Non-odd)} \quad .$$

3.1 Linear Restoring Force

Perhaps the most common type of restoring force used in vibrators is a force which is linear with the displacement. In this case the restoring force $R(x) = x$. In the physical system this means that an increase in the displacement results in a linearly proportional increase in the restoring force. Since $R(x)$ is an odd function the resulting vibration will be about the point $x = 0$. The above type of restoring force is found in many springs when displaced within their elastic limits. In many other springs the deviation of the restoring force from that of a linear spring is so slight that it can be closely approximated by the linear case.

The equation of motion for the above system then reads

$$m\ddot{x} + c_o \operatorname{sgn} \dot{x} + kx = 0 \quad . \tag{3.5}$$

The phase plane differential equation becomes

$$\frac{dy}{dx} = -\omega_o^2\left(\frac{\lambda \operatorname{sgn} y + x}{y}\right) . \tag{3.6}$$

Equation (2.7) yields the expression for y/ω_o which holds for motion during the first half cycle. It is

$$\frac{y}{\omega_o} = -\sqrt{x_1^{\,2} - x^2 - 2\lambda(x_1 - x)} \quad . \tag{3.7}$$

During the second half cycle, we have from equation (2.10)

$$\frac{y}{\omega_o} = \sqrt{x_2^{\,2} - x^2 - 2\lambda(x_2 - x)} \quad . \tag{3.8}$$

Similarly we can write with equation (2.14) the expression which governs motion during the n^{th} half cycle. It is (assuming that $x_1 > x_0$)

$$\frac{y}{\omega_o} = (-1)^n \sqrt{x_n^{\,2} - x^2 + 2(-1)^n \lambda(x_n - x)} \quad . \tag{3.9}$$

The expression $V_o(x)$ defined in equation (3.11) becomes $V_o(x) = (x^2 - x_o^{\,2})/2$. A graph of the function $V_o(x) = (x^2 - x_o^{\,2})/2$ is shown in the Appendix in Figure A-1*.

From equation (3.7) we see that during the first half cycle the phase trajectory is circular in the $(x, y/\omega_o)$ plane since we can re-write the equation (3.7) as

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*In all the graphs of the functions $V_o(x)$ which will appear in this text the equilibrium position will be taken as zero since this is the only equilibrium position for most of the systems discussed and is one of many for the rest of the systems.

$$(x - \lambda)^2 + \left(\frac{y}{\omega_o}\right)^2 = (x_1 - \lambda)^2 \quad . \tag{3.10}$$

This is the equation of a circle with radius $r_1 = x_1 - \lambda$ and center at the point $(\lambda, 0)$. We can solve equation (3.10) for the amplitude $x = x_2$ directly since at this point $y = 0$. Solutions to equation (3.10) with $x = x_2$ are $x_2 = x_1$, and $x_2 = 2\lambda - x_1$. Thus, with initial amplitude x_1, and damping λ, the mass moves to an amplitude x_2 which is 2λ less than its initial value.

Similarly for motion in the second half cycle we can rewrite equation (3.3) as

$$(x + \lambda)^2 + \left(\frac{y}{\omega_o}\right)^2 = (x_2 + \lambda)^2 \quad . \tag{3.11}$$

This is an equation of a circle whose center is at the point $(-\lambda, 0)$ and whose radius is $r_2 = \left| x_2 + \lambda \right|$. We can also express this radius in terms of the initial displacement as $r_2 = x_1 - 3\lambda$. Continuing as above we find

$$\begin{aligned} x_2 &= -x_1 + 2\lambda \\ x_3 &= +x_1 - 4\lambda \\ x_4 &= -x_1 + 6\lambda \quad . \end{aligned}$$

We can write a general expression for the successive amplitudes as a function of the initial displacement and the damping which holds for a linear restoring force as follows:

$$x_n = (-1)^{n-1}\, x_1 + (-1)^n\, (2n - 2)\lambda \tag{3.12}$$

where $n = 1, 2, 3, \ldots N+1$. Thus, we see that consecutive amplitudes

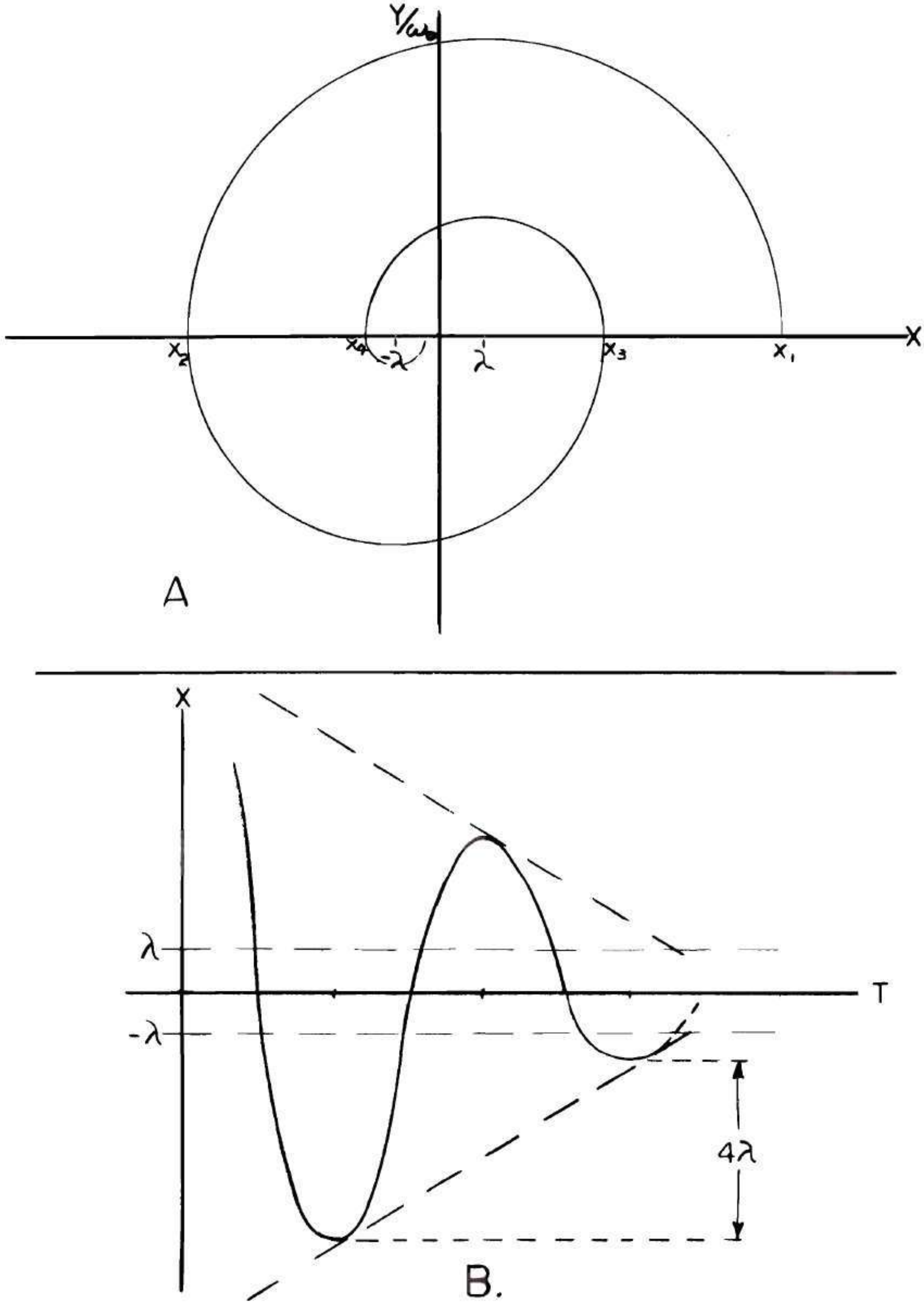


Figure 3.3. Phase Plane and Vibration Envelope for a System with a Linear Restoring Force and Constant Coulomb Damping.

differ by 2λ, or the difference in amplitude between any two consecutive cycles is 4λ. This means that the envelope of vibration is composed of two straight lines which is a well known result. The construction of a phase trajectory showing the preceding information is shown in Figure 3.3A. The curve in Figure 3.3B shows the successive amplitudes of vibration and the resulting straight line envelope.

The expression for $t(x)$ which applies during the n^{th} half cycle of vibration is found by integration of equation (3.9). We have

$$t(x) = \frac{(-1)^n}{\omega_o} \int_{x_n}^{x} \frac{d\eta}{\sqrt{x_n^{\;2} - \eta^2 + 2(-1)^n \lambda(x_n - \eta)}} \tag{3.13}$$

In order to integrate this equation we make the substitution

$$\xi = \frac{\eta + (-1)^n \lambda}{x_n + (-1)^n \lambda} \quad .$$

With the above substitution equation (3.13) becomes

$$t(x) = \frac{(-1)^n}{\omega_o} \int_{1}^{\frac{x + (-1)^n \lambda}{x_n + (-1)^n \lambda}} \frac{d\xi}{\sqrt{1 - \xi^2}} \quad . \tag{3.14}$$

Integrating (3.14) we obtain

$$t(x) = \frac{(-1)^n}{\omega_o} \left[\sin^{-1}\left(\frac{x + (-1)^n \lambda}{x_n + (-1)^n \lambda}\right) - \frac{\pi}{2}\right] \tag{3.15}$$

$$t(x) = \frac{(-1)^n}{\omega_o}\left[\cos^{-1}\left(\frac{x + (-1)^n \lambda}{x_n + (-1)^n \lambda}\right)\right] \quad .$$

Taking the inverse of the above equation gives the expression for x(t) that applies to motion during the n^{th} half cycle. We thus obtain

$$x(t) = \left(x_n + (-1)^n \lambda\right) \cos (\omega_o t) - (-1)^n \lambda \quad . \qquad (3.16)$$

From the above equation we find that for the first half cycle the motion can be described by the equation

$$x(t) = \lambda + (x_1 - \lambda) \cos \omega_o t \quad . \qquad (3.17)$$

Similarly, the motion in the second half cycle can be written as

$$x(t) = - \lambda + (x_2 + \lambda) \cos \omega_o t \quad , \qquad (3.18)$$

which may be expressed as

$$x(t) = - \lambda + (3\lambda - x_1) \cos \omega_o t \quad . \qquad (3.19)$$

Equation (3.16) can be rewritten in terms of the initial displacement as

$$x(t) = (-1)^{n+1} \lambda + \left[(-1)^{n+1} x_1 + (-1)^n (2n-1) \lambda \right] \cos \omega_o t \quad . \qquad (3.20)$$

The total time required for the n^{th} half cycle of vibration is found by integrating from x_n to x_{n+1}. This time is

$$t_n = \frac{(-1)^{n-1}}{\omega_o} \left[\cos^{-1}\left(\frac{x_{n+1} + (-1)^n \lambda}{x_n + (-1)^n \lambda}\right)\right] \quad . \tag{3.21}$$

If motion continues for N half cycles the total time elapsed will be

$$t = -\sum_{n=1}^{N} \frac{(-1)^n}{\omega_o} \left[\cos^{-1}\left(\frac{x_{n+1} + (-1)^n \lambda}{x_n + (-1)^n \lambda}\right)\right] \quad . \tag{3.22}$$

The system will cease to vibrate after N cycles if at that time the restoring force is equal to or less in absolute value than λ. This means that if the motion stops at the end of the N^{th} half cycle $\left|x_{N+1}\right| \le \lambda$, or

$$\left|(-1)^N x_1 + (-1)^{N+1} (2N)\lambda\right| \le \lambda \quad . \tag{3.23}$$

The above equation can be solved for N, i.e., $N \ge \frac{x_1}{2\lambda} - \frac{1}{2}$ where N is the smallest integer that satisfies the inequality.

3.2 Parabolic Restoring Force

In this section we will consider a restoring force of the form $R(x) = x^m \operatorname{sgn} x$ where m is an integer greater than or equal to zero.

From equation (2.1) we obtain the equation governing the motion of the vibrator. It is

$$m\ddot{x} + c_o \operatorname{sgn}\dot{x} + K\left|x\right|^m \operatorname{sgn} x = 0 \quad . \tag{3.24}$$

The differential equation of the phase plane (2.3) then becomes

$$\frac{dy}{dx} = \frac{-\omega_o^{\,2}(\lambda \text{ sgn } y + \left| x \right|^m \text{ sgn } x)}{y} \quad . \tag{3.25}$$

Following the method outlined in Section II we find the expression for the phase trajectories as follows. The equation of the phase trajectory which applies to motion in the first half cycle, obtained from equation (2.7), is

$$\frac{y}{\omega_o} = -\sqrt{2\left[-\lambda(x_1 - x) + \frac{\left|x_1\right|^{m+1} - \left|x\right|^{m+1}}{m+1}\right]} \quad . \tag{3.26}$$

*

Similarly, we find that the equation which applies to the motion in the second half cycle is

$$\frac{y}{\omega_o} = \sqrt{2\left[\lambda(x_2 - x) + \frac{\left|x_2\right|^{m+1} - \left|x\right|^{m+1}}{m+1}\right]} \quad . \tag{3.27}$$

During the third half cycle we have from equation (2.11)

$$\frac{y}{\omega_o} = -\sqrt{2\left[-\lambda(x_3 - x) + \frac{\left|x_3\right|^{m+1} - \left|x\right|^{m+1}}{m+1}\right]} . \tag{3.28}$$

---

* The term $\left|x\right|^m$ sgn x when integrated yields $\frac{\left|x\right|^{m+1}}{m+1}$, since for m odd we can write $\left|x\right|^m$ sgn x as x^m which when integrated is $\frac{x^{m+1}}{m+1}$, or, $\frac{\left|x\right|^{m+1}}{m+1}$, and when m is even the term $\left|x\right|^m$ sgn x or its equivalent, x^m sgn x (an odd function) integrates to $\frac{x^{m+1} \text{ sgn } x}{m+1}$, or, $\frac{\left|x\right|^{m+1}}{m+1}$ (an even function).

We can write the expression for $\frac{y}{\omega_o}$ which applies to the motion during the n^{th} half cycle from (2.14) as follows:

$$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left[(-1)^n \lambda (x_n - x) + \frac{\left|x_n\right|^{m+1} - \left|x\right|^{m+1}}{m + 1} \right]} . \quad (3.29)$$

The above equations are written on the assumption that the initial amplitude is positive. The velocity is consequently negative during the first half cycle. The above equations reduce to the results of section 3.1, where the restoring force was linear, in the case for which $m = 1$, as would be expected.

The time required for the n^{th} half cycle of vibration can be found by integration of equation (3.29) between x_n and x_{n+1}. We find

$$t_n = \frac{(-1)^n}{\omega_o} \int_{x_n}^{x_{n+1}} \frac{dx}{\sqrt{2\left[(-1)^n \lambda(x_n - x) + \frac{\left|x_n\right|^{m+1} - \left|x\right|^{m+1}}{m + 1} \right]}} . \quad (3.30)$$

We find from the definition of $V_o(x)$ given in equation (3.2) that with a parabolic restoring characteristic

$$V_o(x) = \frac{\left|x\right|^{m+1}}{m + 1} - \frac{\left|x_o\right|^{m+1}}{m + 1} . \quad (3.31)$$

The curves $V_o(x)$ are parabolas originating at the point $x = 0$ for $m = 0, 1, 2, 3, \ldots$.

In the following, the results of a parabolic restoring force are shown for the cases $m = 0, 2, 3,$ and 4.

3. 2. 1 Case 1: R(x) = sgn x

The equation of motion becomes

$$m\ddot{x} + c_o \operatorname{sgn} \dot{x} + k \operatorname{sgn} x = 0 \quad . \tag{3.32}$$

This type of restoring force is present when a mass slides on the inclined surfaces shown in Figure 3.4. The restoring force in the

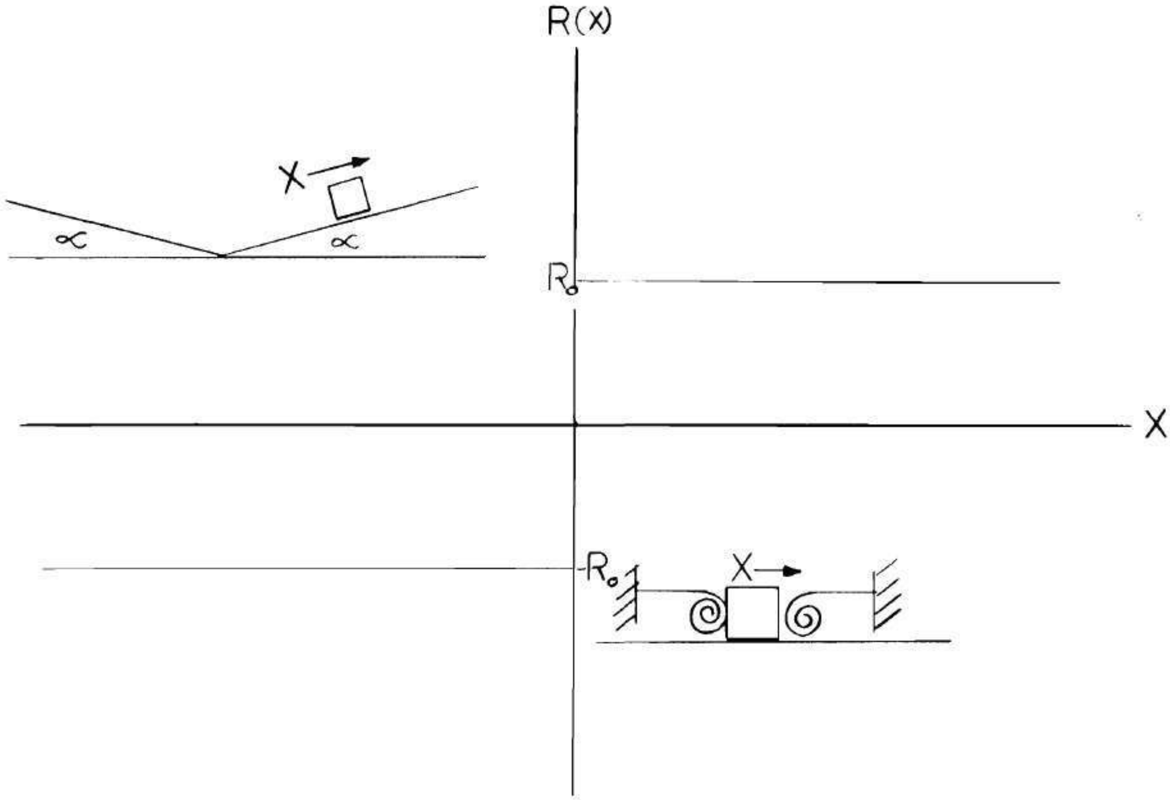


Figure 3.4 Typical Devices with Restoring Forces of the Form R(x) = sgn x.

vibrator shown in Figure 3.4 is written as $r(x) = mg \sin \alpha = k \operatorname{sgn} x$, where the displacement x is measured along the plane.

Some springs of the "Negator" type have a constant restoring characteristic. A vibrator constructed with one "Negator" spring on each side of the mass such that each acts alone over one quarter of a cycle would be of the form considered above. A wire carrying a D.C. current in a magnetic field also could be made to vibrate as described in equation (3.32) if the current or the magnetic field were reversed each time the wire crossed the equilibrium position.

The equation of the phase trajectory that applies to motion during the n^{th} half cycle of vibration follows from (3.29). It is

$$\frac{y}{\omega_o} = (-1)^n \sqrt{2 \left[(-1)^n \lambda \, (x_n - x) + \left| x_n \right| - \left| x \right| \right]} \quad . \quad (3.33)$$

We see immediately from equation (3.32) that motion is possible only for ratios $\lambda = c_o/k$ which satisfy the inequality $0 \leq \lambda < 1$. For $\lambda = 1$, the damping force equals the restoring force and motion will not be possible for any initial displacement, and, if λ is zero, motion once started continues indefinitely.

The curve $V_o(x) = x$ is simply two straight lines (shown in the appendix in Figure A-1) rising at 45 degrees on each side of the origin. From equation (3.33) and Figure A-1 we can find expressions for the unknown amplitudes $x_2, x_3, x_4, \ldots x_{n+1}$ in terms of the initial displacement. At each end point the expression under the radical in (3.33) is zero, i.e., $y = 0$. Thus,

$$(-1)^n \lambda(x_n - x_{n+1}) = \left| x_{n+1} \right| - \left| x_n \right| \quad . \tag{3.34}$$

During the first half cycle we have the two solutions $x_2 = x_1$, and $x_2 = \left[\frac{-(1-\lambda)}{1+\lambda}\right] x_1$, the second solution being the desired unknown amplitude. Similarly, during the second half cycle we find $x_3 = x_2$, and $x_3 = \left[\frac{-(1-\lambda)}{1+\lambda}\right] x_2$, or, writing x_2 in terms of x_1, we have $x_3 = \left[\frac{-(1-\lambda)}{1+\lambda}\right]^2 x_1$. In general, the unknown amplitudes can be found from the equation

$$x_{n+1} = \left[\frac{-(1-\lambda)}{(1+\lambda)} \right]^n x_1 \quad . \tag{3.35}$$

Notice that if the damping force is small enough for vibrations to take place, i.e., $0 \leq \lambda < 1$, that motion once started will theoretically take an infinite time to die out. Each amplitude is reduced by the ratio $(1-\lambda)/(1+\lambda)$ times the preceding amplitude, thus, the amplitudes approach zero in an infinite time. Notice that in this case motion does not stop at a finite amplitude which satisfies the inequality $\left| R(x)_{N+1} \right| \leq \lambda$ since the restoring force $R(x) = \text{sgn } x$ will always be the larger than λ if vibration is possible (i.e., $0 \leq \lambda < 1$).

3. 2. 2 Case 2: $R(x) = x^2 \text{ sgn } x$

The equation of motion of the vibrator becomes

$$m\ddot{x} + c_o \text{ sgn } \dot{x} + k\, x^2 \text{ sgn } x = 0 \quad . \tag{3.36}$$

The phase plane equation (3.29) can be written in a general form which applies to motion during the n^{th} half cycle. It is

$$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left[(-1)^n\, 2\, [(-1)^n\, \lambda(x_n - x) + \frac{|x_n|^3 - |x|^3}{3}\,]\right]} \; . \qquad (3.37)$$

The function $V_o(x) = \frac{x^3 \text{ sgn } x}{3}$ is plotted in Figure A-1, for use in finding successive amplitudes. It can be seen from the restoring force that $\left|R_{(N+1)}\right| \le \lambda$ when $x^2_{N+1} \le \lambda$. Thus, when one of the amplitudes satisfies this inequality the vibrator will remain at rest.

3. 2. 3 Case 3: $R(x) = x^3$

The equation of motion becomes

$$m\ddot{x} + c_o \text{ sgn } \dot{x} + k\, x^3 = 0 \quad . \qquad (3.38)$$

The phase plane equation (3.39) which applies to motion in the n^{th} half cycle is

$$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left[(-1)^n\, \lambda(x_n - x) + \frac{x_n^{\,4} - x^4}{4}\right]} \quad . \qquad (3.39)$$

A curve of the function $V_o(x) = \frac{x^4}{4}$ is shown in Figure A-1. Successive amplitudes are found as previously outlined and the vibrator will come to rest when $\left|x_{N+1}\right|^3 \le \lambda$.

3. 2. 4 Case 4: $R(x) = x^4 \text{ sgn } x$

The equation of motion reads

$$m\ddot{x} + c_o \text{ sgn } \dot{x} + k\, x^4 \text{ sgn } x = 0 \quad , \qquad (3.40)$$

and the phase trajectories are described by the equation

$$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left[(-1)^n \lambda(x_n - x) + \frac{|x_n|^5 - |x|^5}{5}\right]} \ . \tag{3.41}$$

A graph of the function $V_o(x) = \frac{x^5 \operatorname{sgn} x}{5}$ is shown in Figure A-1. The vibrator will come to rest for any amplitude x_{N+1} which satisfies the condition $x^4_{N+1} \leq \lambda$.

3.3 Combinations of Parabolic Restoring Force

We concern ourselves in this section with a restoring force of the form

$$R(x) = (|x|^m + \epsilon |x|^p) \operatorname{sgn} x$$

where ϵ is a constant which may be positive, negative, or zero. The values of m and p must be greater than or equal to zero, to avoid an infinite restoring force at the point $x = 0$. In many physical systems it is found that the restoring force is not completely of one form or another. For example, the spring may be essentially linear with a small higher order component. Systems of many forms might well be approximated by a combination of restoring forces with the constant and the powers being adjusted to best fit the physical system. Using equation (2.7) or (3.2) we can write the expression governing motion during the first half cycle, where we assume the velocity is negative. It is

$$\frac{y}{\omega_o} = -\sqrt{2\left[-\lambda(x_1 - x) + \frac{x_1^{m+1} - x^{m+1}}{m+1} + \epsilon \frac{x_1^{p+1} - x^{p+1}}{p+1}\right]} . \tag{3.42}$$

Similarly from (2.9) the phase plane equation for the second half cycle can be written as

$$\frac{y}{\omega_o} = \sqrt{2\left[\lambda\,(x_2 - x) + \frac{\left|x_2\right|^{m+1} - \left|x\right|^{m+1}}{m+1} + \epsilon\,\frac{\left|x_2\right|^{p+1} - \left|x\right|^{p+1}}{p+1}\right]}\,. \tag{3.43}$$

The equation that applies to the motion during the n^{th} half cycle follows from (2.14). It is

$$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left[(-1)^n\,(x_n - x)\,\lambda + \frac{\left|x_n\right|^{m+1} - \left|x\right|^{m+1}}{m+1} + \epsilon\,\frac{\left|x_n\right|^{m+1} - \left|x\right|^{p+1}}{p+1}\right]}\,. \tag{3.44}$$

From the definition of the function $V_o(x)$ given in (2.12) we see that

$$V_o(x) = \frac{\left|x\right|^{m+1}}{m+1} + \epsilon\,\frac{\left|x\right|^{p+1}}{p+1} \tag{3.45}$$

with $x_o = 0$ as the equilibrium position.

It should be remembered that the results given above are general and can be applied to any system with constant Coulomb damping and a combination of parabolic restoring forces. The results are extended to cover any type of polynomial characteristic which is an odd function of displacement as follows.

The expression governing the motion in the n^{th} half cycle is given as equation (3.46) for a restoring force of the form

$$R(x) = \left(\left|x\right| + a_1\left|x\right|^2 + a_2\left|x\right|^3 + \ldots + a_{j-1}\left|x\right|^j\right)\,\text{sgn}\,x$$

where a_1, a_2, a_3, . . . a_j are constants. It is

$$\frac{y}{\omega_o} = (-1)^n \left\{ 2 \left[(-1)^n \lambda(x_n - x) + \frac{|x_n|^2 - |x|^2}{2} + a_1 \frac{|x_n|^3 - |x|^3}{3} + \ldots\ldots + a_{j-1} \frac{|x_n|^{j+1} - |x|^{j+1}}{j+1} \right] \right\}^{\frac{1}{2}} . \quad (3.46)$$

The ability to add components of many powers enables the reader to fit many spring types with a restoring characteristic of the polynomial type. In the paragraphs that follow we consider an important special case of the above type of restoring force where the powers m and p are taken as one and three respectively.

3. 3. 1 Duffing Type Restoring Force

The Duffing type restoring force is written as $R(x) = x + \epsilon\, x^3$, which makes the equation of motion of the form

$$m\ddot{x} + c_o \operatorname{sgn} \dot{x} + k\,(x + \epsilon\, x^3) = 0 \quad . \quad (3.47)$$

The equation governing the motion during the n^{th} half cycle of motion is (from 3.46)

$$\frac{y}{\omega_o} = (-1)^n \sqrt{2 \left[(-1)^n \lambda(x_n - x) + \frac{x_n^2 - x^2}{2} + \epsilon \frac{x_n^4 - x^4}{4} \right]} . \quad (3.48)$$

A graph of the function $V_o(x) = \frac{x^2}{2} + \epsilon \frac{x^4}{4}$ is given in Figure A-2 for use in finding successive amplitudes of vibration. The curves are for $\epsilon = \pm\, 0.1$ and for $\epsilon = \pm\, 0.05$. This corresponds to restoring forces of both the "hardening" ($\epsilon > 0$) and the "softening" ($\epsilon < 0$) type.

Successive amplitudes of vibration can be found by the method outlined in Figure 3.1. Motion will cease when one of the amplitudes satisfies the inequality $\left| x_{N+1} + \epsilon \, x^3_{N+1} \right| \leq \lambda$.

Since we have assumed the system is "restored", i.e., $xr(x) > 0$, the initial amplitude x_1 has to be limited for vibrators where $\epsilon < 0$.

3.4 Sinusoidal Restoring Force

In this section the restoring force $R(x) = \sin x$ where x is a measure of angle or displacement in radians. Vibrators of the pendulum type (Figure 2.1) have a restoring force of the above form. The equation of motion for the vibrator in Figure 2.1 is

$$m\ddot{x} + c_o \operatorname{sgn} \dot{x} + k \sin x = 0 \quad . \tag{3.49}$$

Thus, motion during the first half cycle can be described with equation (2.14) as

$$\frac{y}{\omega_o} = - \sqrt{2 \left[- \lambda(x_1 - x) + \cos x - \cos x_1 \right]} \quad . \tag{3.50}$$

Motion during the second half cycle can be expressed as

$$\frac{y}{\omega_o} = \sqrt{2 \left[\lambda(x_2 - x) + \cos x - \cos x_2 \right]} \quad . \tag{3.51}$$

A general expression which describes motion during the n^{th} half cycle can be written as

$$\frac{y}{\omega_o} = (-1)^n \sqrt{2 \left[(-1)^n \, (x_n - x) + \cos x - \cos x_n \right]} \, . \tag{3.52}$$

A graph of $\overline{V}_o(x) = 1 - \cos x$ is shown in Figure A-3. We notice that for vibration to take place about the equilibrium position $x_o = 0$, the initial amplitude must be restricted to the range $-\pi \le x_1 \le \pi$. Since no vibration can begin if the restoring force is less than the damping force, initial amplitudes are restricted to those that satisfy the inequality $\left| \sin x_1 \right| > \lambda$. In the physical system shown in Figure 2.1 the above restriction prohibits initial displacements "near" $\pm \pi$ radians, and states that vibration will stop when one of the successive amplitudes satisfies the inequality,

$$\left| \sin x_{N+1} \right| \le \lambda \quad .$$

3.5 Hyperbolic Tangent Restoring Force

If the restoring force is of the form $R(x) = \tanh x$, the equation of motion becomes

$$m\ddot{x} + c_o \operatorname{sgn} \dot{x} + k \tanh x = 0 \quad . \tag{3.53}$$

The motion during the first half cycle can then be described by the equation

$$\frac{y}{\omega_o} = - \sqrt{2 \left[- \lambda(x_1 - x) + \ell n\left(\frac{\cosh x_1}{\cosh x}\right) \right]} \quad . \tag{3.54}$$

Similarly, for motion during the second half cycle we have

$$\frac{y}{\omega_o} = \sqrt{2 \left[\lambda(x_2 - x) + \ell n\left(\frac{\cosh x_2}{\cosh x}\right) \right]} \quad , \tag{3.55}$$

or in general,

$$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left[(-1)^n \lambda(x_n - x) + \ell n\left(\frac{\cosh x_n}{\cosh x}\right)\right]} \quad . \tag{3.56}$$

The graph of $\overline{V}_o(x) = \ell n(\cosh x)$ is shown in Figure A-3. Amplitudes can be obtained by the graphical method of Figure 3-1.

3.6 Non-odd Restoring Force

Consider as an example the non-odd restoring force $R(x) = x + \epsilon\, x^2$, where ϵ is a constant. A graph for the above restoring force is shown in Figure 3-2. The equation governing the motion then becomes

$$m\ddot{x} + c_o \operatorname{sgn} \dot{x} + k\,(x + \epsilon\, x^2) = 0 \quad . \tag{3.57}$$

The equation governing motion during the first half cycle can be written (assuming $x_1 > x_o$) as

$$\frac{y}{\omega_o} = -\sqrt{2\left[-\lambda(x_1 - x) + \frac{x_1^2 - x^2}{2} + \epsilon\,\frac{(x_1^3 - x^3)}{3}\right]} \quad . \tag{3.58}$$

During the second half cycle of motion we have

$$\frac{y}{\omega_o} = \sqrt{2\left[\lambda(x_2 - x) + \frac{x_2^2 - x^2}{2} + \epsilon\,\frac{(x_2^3 - x^3)}{3}\right]} \, . \tag{3.59}$$

Thus, we can write the expression for $\frac{y}{\omega_o}$ which holds for the n^{th} half cycle as

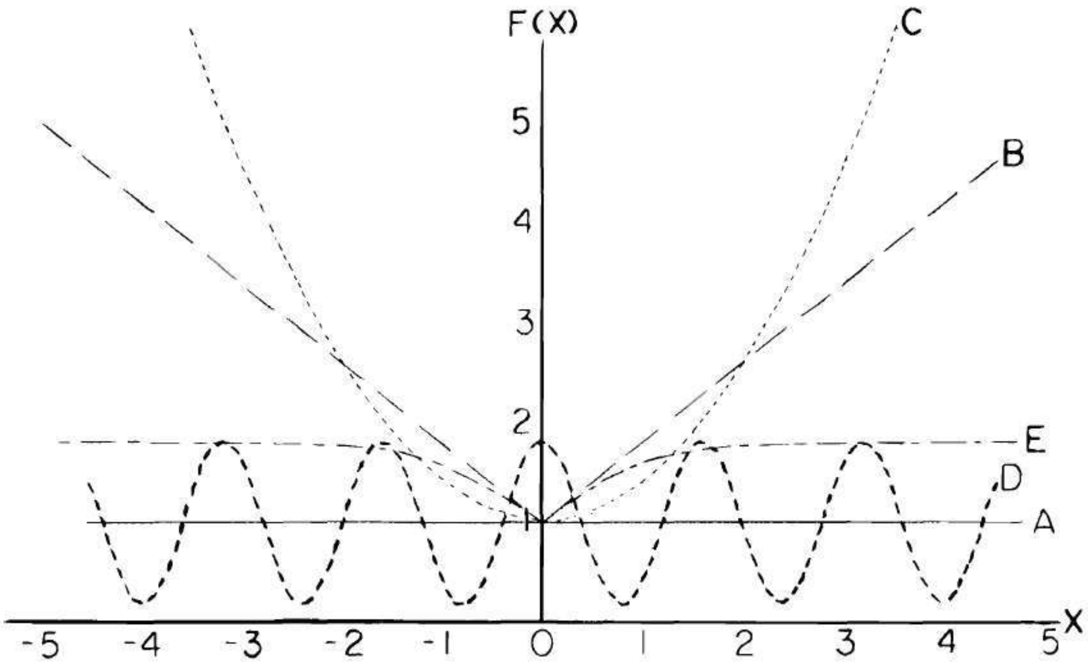
$$\frac{y}{\omega_o} = (-1)^n \sqrt{2 \left[\lambda (-1)^n (x_n - x) + \frac{x_n^2 - x^2}{2} + \epsilon \frac{(x_n^3 - x^3)}{3} \right]} . \quad (3.60)$$

Equation (3.60) approximates that of a vibrator with a linear spring when ϵ is small. Curves of $V_o(x)$ have been drawn in Figure A-4 for $\epsilon = \pm 0.2$ and $\epsilon = \pm 0.4$. It can be seen that for large ϵ the initial amplitudes have to be reduced for a given damping to insure that the system will oscillate about $x_o = 0$. For example, if $R(x) = x + 0.4 x^2$ and $\lambda = 0.5$, the initial amplitude must lie approximately between -2.4 and $+2$. If λ is increased the permissible initial amplitudes also increase. Amplitudes can be obtained by the method of Figure 3-1 and vibration will cease when $\left| x_{N+1} + \epsilon x_{N+1}^2 \right| \leq \lambda$.

CHAPTER IV

DISPLACEMENT DEPENDENT COULOMB DAMPING

In this chapter the results of the preceding chapter will be extended to cover Coulomb damping which depends upon displacement. Thus, the damping force will be $d(x,\dot{x}) = c_o f(x)\ D(x) = c_o f(x)\ \mathrm{sgn}\ \dot{x}$, as introduced in Chapter II. The governing equations for the non-negative functions will be developed in the following paragraphs. The results for $f(x) = 1$ are included for comparison.



A. $f(x) = 1$

B. $f(x) = 1 + \alpha|x|$

C. $f(x) = 1 + \alpha x^2$

D. $f(x) = 1 + \alpha \cos 4x$

E. $f(x) = 1 + \alpha|\tanh x|$

Figure 4.1 Non-Negative Damping Forces.

α is a positive constant less than or equal to one. Graphs of the above functions are shown in Figure 4.1 for $\alpha = 0.8$ (except curve C which is for $\alpha = 0.4$).

A graphical method for finding successive amplitudes of vibration was developed in Chapter III for the special case $f(x) = 1$. We now extend the method previously given to cover arbitrary non-negative functions $f(x)$.

Equation (2.10) is rewritten for convenience

$$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left[(-1)^n \lambda \int_x^{x_n} f(\eta)\, d\eta + \int_x^{x_n} R(\eta)\, d\eta\right]} \ . \qquad (4.1)$$

We will introduce an overlay method for finding successive amplitudes by defining an overlay function $L(x,n)$ as

$$L(x,n) = (-1)^n \lambda \int_{x_o}^{x} f(\eta)\, d\eta \qquad (4.2)$$

Using the definition of $V_o(x)$ as previously given and (4.2), we have from equation (4.1)

$$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left[L(x_n,n) - L(x,n) + V_o(x_n) - V_o(x)\right]} \ . \qquad (4.3)$$

At $x = x_n$ the velocity is zero and at $x = x_{n+1}$ the velocity again becomes zero.

Thus, we can write the equation

$$L(x_n,n) - L(x_{n+1},n) + V_o(x_n) - V_o(x_{n+1}) = 0 \quad . \tag{4.4}$$

In some cases equation (4.4) can be solved for x_{n+1}. For example, if $f(x) = 1$ and $R(x) = x$, we have from (4.4)

$$(-1)^n \lambda x_n - (-1)^n \lambda x_{n+1} + \frac{x_n^2 - x_{n+1}^2}{2} = 0 \quad ; \tag{4.5}$$

thus,

$$x_{n+1} = (-1)^{n+1} 2\lambda - x_n \quad . \tag{4.6}$$

Similarly, if $f(x) = 1 + |x|$ and $R(x) = x$, we can write from (4.4)

$$(-1)^n \lambda\left(x_n - x_{n+1} + \frac{x_n \operatorname{sgn} x_n - x_{n+1} \operatorname{sgn} x_{n+1}}{2}\right) + \frac{x_n^2 - x_{n+1}^2}{2} = 0 . \tag{4.7}$$

Since the initial amplitude x_1 was assumed to be positive we can write the product $(-1)^n \operatorname{sgn} x_n$ as -1 and the product $(-1)^n \operatorname{sgn} x_{n+1}$ as $+1$ for any n. Equation (4.7) can now be rewritten as

$$x_{n+1}^2 + \lambda[1 + (-1)^n 2]\, x_{n+1} + \lambda(1 - (-1)^n 2)\, x_n - x_n^2 = 0 \tag{4.8}$$

whose solution is

$$x_{n+1} = -\frac{\lambda}{2}(1 + (-1)^n 2) \pm \sqrt{\frac{\lambda^2(1 + 4[1 + (-1)^n])}{4} + x_n^2 - x(1 - (-1)^n 2)\, x_n} \quad . \tag{4.9}$$

In general, however, equation (4.4) cannot be solved analytically for x_{n+1}. In these cases a graphical method will prove very useful. The solution to (4.4) involves subtracting from the term $V_o(x_n) - V_o(x)$ the term $L(x,n) - L(x_n,n)$.

The graphical solution of equation (4.4) can be done as outlined below and pictured in Figure 4.2. First, a curve of $V_o(x)$ is drawn, and on a separate sheet of transparent paper the curve $L(x,1)$ is drawn. The overlay is then placed on the curve $V_o(x)$ such that the ordinates of each curve are superimposed. The overlay is then shifted up or down on the $V_o(x)$ axis until the curves intersect at the point $[x_1, V_o(x_1)]$. The other intersection of the curves then gives the unknown amplitude x_2. The overlay is then turned face down since this corresponds to drawing a curve of $L(x,2)$. With the ordinates superimposed the overlay is shifted up or down until it intersects the $V_o(x)$ curve at the point $[x_2, V_o(x_2)]$. The other intersection of the curves then give the amplitude x_3. The next amplitude is found in the same way as the first was found.

The method continues until one of the amplitudes found satisfies the inequality $\left|R(x_{n+1})\right| \leq \lambda f(x_{n+1})$. When the above inequality is satisfied the mass cannot begin to move since the restoring force is less than the available damping force.

In the following paragraphs the results of Chapters II and III will be extended as indicated above for the functions.

$$f(x) = 1 + \alpha|x|$$
$$f(x) = 1 + \alpha x^2$$
$$f(x) = 1 + \alpha \cos 4x$$
$$f(x) = 1 + \alpha|\tanh x|$$

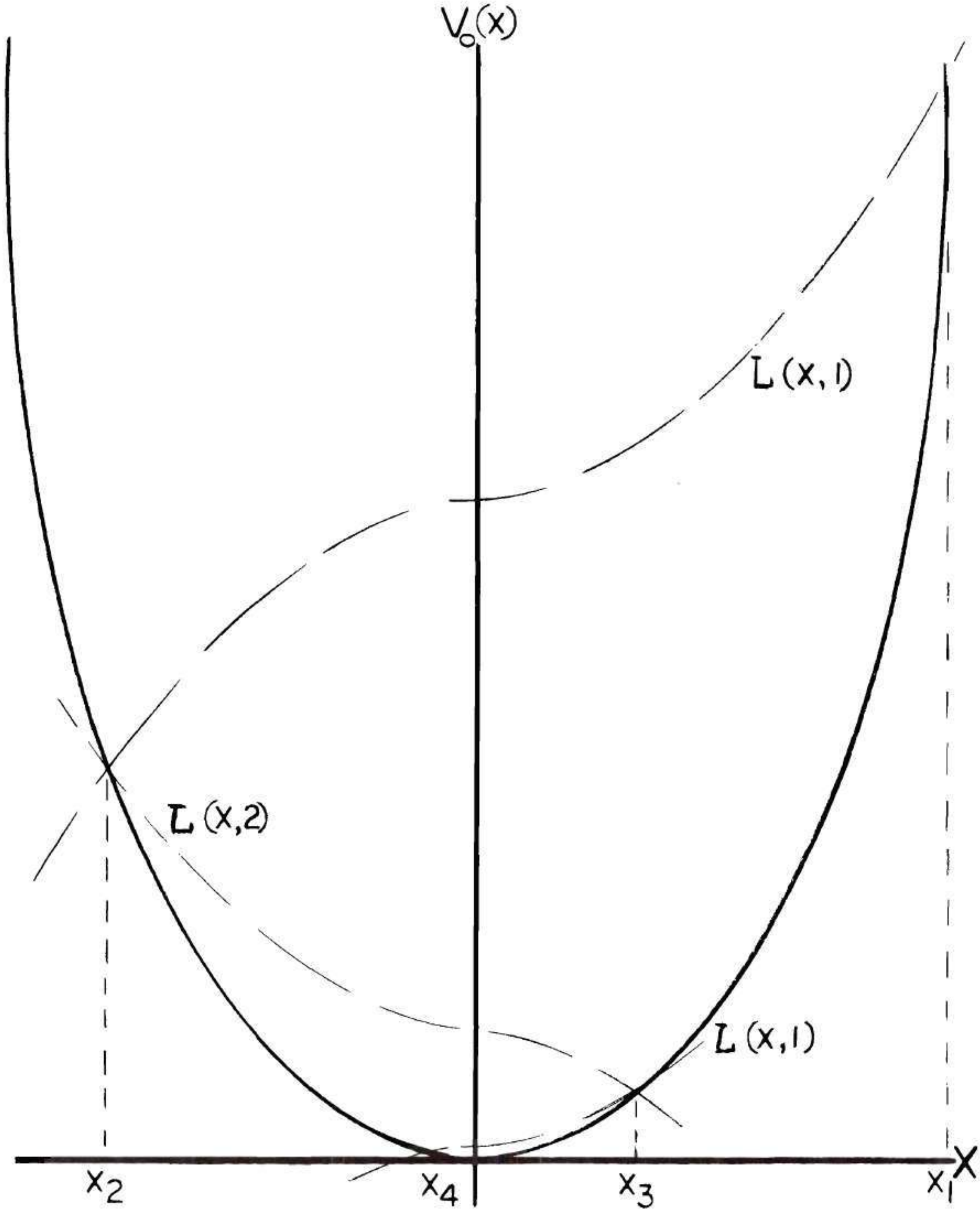


Figure 4.2. Graphical Method of Finding Successive Amplitudes for Systems with Coulomb Damping Which Depends on Displacement.

where α is a positive constant in the range $0 \leq \alpha \leq 1$. Curves showing the above functions appear in Figure 4.1.

4.1 Linear Restoring Force

The equation of motion can be written as

$$m\ddot{x} + c_o\, f(x)\ \text{sgn}\ \dot{x} + k\,x = 0 \quad . \tag{4.10}$$

One form of vibrator of the above type appears in Figure (2.1). The differential equation of the motion in the phase plane is

$$\frac{dy}{dx} = -\,\omega_o^{\,2}\left(\frac{\lambda\, f(x)\ \text{sgn}\ y + x}{y}\right) . \tag{4.11}$$

From equation (2.7) we have the motion in the first half cycle described by

$$\frac{y}{\omega_o} = -\sqrt{2\left[\,-\,\lambda\int_x^{x_1} f(\eta)\ d\eta + \frac{x_1^{\,2} - x^2}{2}\,\right]} \quad . \tag{4.12}$$

Similarly, from equation (2.10) we find the motion in the second half cycle described by

$$\frac{y}{\omega_o} = \sqrt{2\left[\,\lambda\int_x^{x_2} f(\eta)\ d\eta + \frac{x_2^{\,2} - x^2}{2}\,\right]} \quad . \tag{4.13}$$

The motion during the n^{th} half cycle can be described by

$$\frac{y}{\omega_o} = (-1)^n\sqrt{2\left[\,(-1)^n\,\lambda\int_x^{x_n} f(\eta)\ d\eta + \frac{x_n^{\,2} - x^2}{2}\,\right]} \ , \tag{4.14}$$

$$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left[L(x_n) - L(x) + V_o\,(x_n) - V_o\,(x) \right]} \quad . \tag{4.15}$$

Table 4.1 gives equations of motion in the phase plane, and the location of curves to be used in the graphical solution for functions f(x) previously listed.

4.2 Parabolic Restoring Force

Consider now a restoring force of the form

$$R(x) = \left|x\right|^m \operatorname{sgn} x$$

where m is a non-negative integer.

The equation of motion with R(x) described above reads

$$m\ddot{x} + c_o\, f(x) \operatorname{sgn} \dot{x} + k\left|x\right|^m \operatorname{sgn} x = 0 \quad , \tag{4.16}$$

and the differential equation of the motion in the phase plane becomes

$$\frac{dy}{dx} = \frac{-\,\omega_o^{\,2}\,(\lambda\, f(x) \operatorname{sgn} y + \left|x\right|^m \operatorname{sgn} x)}{y} \quad . \tag{4.17}$$

We have from equation (2.11) an expression for the velocity as a function of displacement during the n^{th} half cycle of motion. It is

$$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left[(-1)^n \lambda \int_x^{x_n} f(\eta)\, d\eta + \frac{\left|x_n\right|^{m+1} - \left|x\right|^{m+1}}{m+1} \right]} \quad . \tag{4.18}$$

Tables 4.2, 4.3, and 4.4 give the equations of motion, phase plane equations and location of $V_o(x)$ and L(x,1) curves. The curves

Table 4.1 Governing Equations for Restoring Force $R(x) = x$

PHASE PLANE EQUATIONS FOR n^{th} HALF CYCLE
FOR MOTION SATISFYING THE EQUATION OF MOTION $m\ddot{x} + c_o\, f(x)\, \mathrm{sgn}\, \dot{x} + k\, R(x) = 0$

$f(x)$		Curves $V_o(x)$	Curves $L(x,1)$
1	$\frac{y}{\omega_o} = (-1)^n \sqrt{x_n^2 - x^2 + (-1)^n\, 2\lambda\, (x_n - x)}$	A-1	A-5
$1+\alpha\lvert x\rvert$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda \left(x_n - x + \frac{\alpha}{2}\left(\mathrm{sgn}\, x_n\, x_n^2 - \mathrm{sgn}\, x\, x^2\right)\right) + \frac{x_n^2 - x^2}{2}\right)}$	A-1	A-6
$1+\alpha x^2$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda \left(x_n - x + \frac{\alpha}{3}\left(x_n^3 - x^3\right)\right) + \frac{x_n^2 - x^2}{2}\right)}$	A-1	A-7 A-8
$1+\alpha\cos 4x$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda \left(x_n - x + \frac{\alpha}{4}\left(\sin 4x_n - \sin 4x\right)\right) + \frac{x_n^2 - x^2}{2}\right)}$	A-1	A-9
$1+\alpha\lvert\tanh x\rvert$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda\left(x_n - x + \alpha\left(\mathrm{sgn}\, x_n\, \ell n(\cosh x_n) - \mathrm{sgn}\, x\, \ell n(\cosh x)\right)\right) + \frac{x_n^2 - x^2}{2}\right)}$	A-1	A-10

Table 4.2 Governing Equations for Restoring Force $R(x) = \operatorname{sgn}(x)$

PHASE PLANE EQUATIONS FOR n^{th} HALF CYCLE
FOR MOTION SATISFYING THE EQUATION OF MOTION $m\ddot{x} + c_o f(x) \operatorname{sgn} \dot{x} + k R(x) = 0$

$f(x)$		Curves $V_o(x)$	Curves $L(x,1)$
1	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda (x_n - x) + \lvert x_n\rvert - \lvert x\rvert\right)}$	A-1	A-5
$1+\alpha\lvert x\rvert$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda \left(x_n - x + \frac{\alpha}{2}(x_n^2 \operatorname{sgn} x_n - x^2 \operatorname{sgn} x)\right) + \lvert x_n\rvert - \lvert x\rvert\right)}$	A-1	A-6
$1+\alpha x^2$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda \left(x_n - x + \frac{\alpha}{3}(x_n^3 - x^3)\right) + x_n - x\right)}$	A-1	A-7 A-8
$1+\alpha\cos 4x$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda \left(x_n - x + \frac{\alpha}{4}(\sin 4x_n - \sin 4x)\right) + \lvert x_n\rvert - \lvert x\rvert\right)}$	A-1	A-9
$1+\alpha\lvert\tanh x\rvert$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda\left(x_n - x + \alpha\left(\operatorname{sgn} x_n \ln(\cosh x_n) - \operatorname{sgn} x \ln(\cosh x)\right)\right) + \lvert x_n\rvert - \lvert x\rvert\right)}$	A-1	A-10

Table 4.3 Governing Equations for Restoring Force $R(x) = x^2 \operatorname{sgn} x$

PHASE PLANE EQUATIONS FOR n^{th} HALF CYCLE

FOR MOTION SATISFYING THE EQUATION OF MOTION $m\ddot{x} + c_o f(x) \operatorname{sgn} \dot{x} + k R(x) = 0$

f(x)		Curves $V_o(x)$	Curves $L(x,1)$
1	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda(x_n - x) + \frac{\lvert x_n\rvert^3 - \lvert x\rvert^3}{3}\right)}$	A-1	A-5
$1+\alpha\lvert x\rvert$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda\left(x_n - x + \frac{\alpha}{2}(x_n^2 \operatorname{sgn} x_n - x^2 \operatorname{sgn} x)\right) + \frac{\lvert x_n\rvert^3 - \lvert x\rvert^3}{3}\right)}$	A-1	A-6
$1+\alpha x^2$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda\left(x_n - x + \frac{\alpha}{3}(x_n^3 - x^3)\right) + \frac{\lvert x_n\rvert^3 - \lvert x\rvert^3}{3}\right)}$	A-1	A-7 A-8
$1+\alpha\cos 4x$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda\left(x_n - x + \frac{\alpha}{4}(\sin 4x_n - \sin 4x)\right) + \frac{\lvert x_n\rvert^3 - \lvert x\rvert^3}{3}\right)}$	A-1	A-9
$1+\alpha\lvert\tanh x\rvert$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda\left(x_n - x + \alpha\left(\operatorname{sgn} x_n \ln(\cosh x_n) - \operatorname{sgn} x \ln(\cosh x)\right)\right) + \frac{\lvert x_n\rvert^3 - \lvert x\rvert^3}{3}\right)}$	A-1	A-10

Table 4.4 Governing Equations for Restoring Force $R(x) = x^3$

PHASE PLANE EQUATIONS FOR n^{th} HALF CYCLE
FOR MOTION SATISFYING THE EQUATION OF MOTION $m\ddot{x} + c_o\, f(x)\, \operatorname{sgn} \dot{x} + k\, R(x) = 0$

f(x)		Curves $V_o(x)$	Curves $L(x,1)$
1	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda\, (x_n - x) + \frac{x_n^4 - x^4}{4}\right)}$	A-1	A-5
$1+\alpha\lvert x\rvert$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda\left(x_n - x + \frac{\alpha}{2}\,(x_n^2 \operatorname{sgn} x_n - x^2 \operatorname{sgn} x)\right) + \frac{x_n^4 - x^4}{4}\right)}$	A-1	A-6
$1+\alpha x^2$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda\left(x_n - x + \frac{\alpha}{3}\,(x_n^3 - x^3)\right) + \frac{x_n^4 - x^4}{4}\right)}$	A-1	A-7 A-8
$1+\alpha\cos 4x$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda\left(x_n - x + \frac{\alpha}{4}\,(\sin 4x_n - \sin 4x)\right) + \frac{x_n^4 - x^4}{4}\right)}$	A-1	A-9
$1+\alpha\lvert\tanh x\rvert$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda\left(x_n - x + \alpha\left(\operatorname{sgn} x_n\, \ell n(\cosh x_n) - \operatorname{sgn} x\, \ell n(\cosh x)\right)\right) + \frac{x_n^4 - x^4}{4}\right)}$	A-1	A-10

are for the parabolic restoring forces discussed in Chapter II, with the damping functions previously considered.

4.3 Combinations of Parabolic Restoring Forces

As was done in section 3.3, we take the restoring force in the form

$$R(x) = (|x|^m + \epsilon(|x|^p) \text{ sgn } x$$

where ϵ is any constant and m and p are non-negative integers. The equation of motion then becomes

$$m\ddot{x} + c_o f(x) \text{ sgn } \dot{x} + k \text{ sgn } x (|x|^m + \epsilon|x|^p) = 0 \quad , \qquad (4.19)$$

and the differential equation describing the motion in the phase plane is

$$\frac{dy}{dx} = - \omega_o^2 \left(\frac{\lambda f(x) \text{ sgn } y + \text{sgn } x (|x|^m + \epsilon|x|^p)}{y} \right) . \qquad (4.20)$$

We can write the phase plane equation which applies to the motion during the first half cycle, where $y < 0$, as

$$\frac{y}{\omega_o} = - \sqrt{2\left(- \lambda \int_x^{x_1} f(\eta)\, d\eta + \frac{|x_1|^{m+1} - |x|^{m+1}}{m + 1} + \epsilon \frac{|x_1|^{p+1} - |x|^{p+1}}{p + 1}\right)} . \qquad (4.21)$$

Similarly, during the second half cycle

$$\frac{y}{\omega_o} = \sqrt{2\left(\lambda \int_x^{x_2} f(\eta)\, d\eta + \frac{|x_2|^{m+1} - |x|^{m+1}}{m + 1} + \epsilon \frac{|x_2|^{p+1} - |x|^{p+1}}{p + 1}\right)} . \quad (4.22)$$

The equation describing motion in the phase plane during the n^{th} half cycle is written

$$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left(\lambda \int_x^{x_n} f(\eta)\, d\eta + \frac{\left|x_n\right|^{m+1} - \left|x\right|^m}{m+1} + \epsilon \frac{\left|x_n\right|^{p+1} - \left|x\right|^{p+1}}{p+1}\right)} . \tag{4.23}$$

The following equations are written for the special case $m = 1$, $p = 3$, where the restoring force takes the form $R(x) = x + \epsilon x^3$. The equation of motion takes the form

$$m\ddot{x} + c_o f(x) \operatorname{sgn} \dot{x} + k(x + \epsilon x^3) = 0 , \tag{4.24}$$

and the motion during the n^{th} half cycle can be described by the equation

$$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left[(-1)^n \lambda \int_x^{x_n} f(\eta)\, d\eta + \frac{x_n^2 - x^2}{2} + \frac{\epsilon}{4}(x_n^4 - x^4)\right]} . \tag{4.25}$$

The results of the combination of various damping functions with a vibrator of the "Duffing" type are shown in Table 4.5.

4.4 Sinusoidal Restoring Force

With a restoring force $R(x) = \sin x$ we can write the equation of motion as

$$m\ddot{x} + c_o f(x) \operatorname{sgn} \dot{x} + k \sin x = 0 . \tag{4.26}$$

The above equation describes the motion of a pendulum with friction which depends upon the position of the mass. For example, a pendulum

Table 4.5 Governing Equations for Restoring Force $R(x) = x + \epsilon x^3$

PHASE PLANE EQUATIONS FOR n^{th} HALF CYCLE
FOR MOTION SATISFYING THE EQUATION OF MOTION $m\ddot{x} + c_o f(x) \operatorname{sgn} \dot{x} + k R(x) = 0$

$f(x)$		Curves $V_o(x)$	Curves $L(x,1)$
1	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda (x_n - x) + \frac{x_n^2 - x^2}{2} + \epsilon \frac{x_n^4 - x^4}{4}\right)}$	A-2	A-5
$1+\alpha\lvert x\rvert$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left(\lambda \left(x_n - x + \frac{\alpha}{2}(\operatorname{sgn} x_n\, x_n^2 - \operatorname{sgn} x\, x^2)\right) + \frac{x_n^2 - x^2}{2} + \epsilon \frac{x_n^4 - x^4}{4}\right)}$	A-2	A-6
$1+\alpha x^2$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda\left(x_n - x + \frac{\alpha}{3}(x_n^3 - x^3)\right) + \frac{x_n^2 - x^2}{2} + \epsilon \frac{x_n^4 - x^4}{4}\right)}$	A-2	A-7 A-8
$1+\alpha\cos 4x$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda \left(x_n - x + \frac{\alpha}{4}(\sin 4x_n - \sin 4x)\right) + \frac{x_n^2 - x^2}{2} + \epsilon \frac{x_n^4 - x^4}{4}\right)}$	A-2	A-9
$1+\alpha\lvert\tanh x\rvert$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda\left(x_n - x + \alpha\left(\operatorname{sgn} x_n\, \ln(\cosh x_n) - \operatorname{sgn} x\, \ln(\cosh x)\right)\right) + \frac{x_n^2 - x^2}{2} + \epsilon \frac{x_n^4 - x^4}{4}\right)}$	A-2	A-10

swinging inside a cylinder with a surface of variable roughness could vibrate according to the above equation.

The equation of motion which applies during the n^{th} half cycle is

$$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda \int_x^{x_n} f(\eta)\, d\eta + \cos x - \cos x_n\right)} . \qquad (4.27)$$

Table 4.6 summarizes the results of a vibrator with a sinusoidal restoring force for various damping functions.

4.5 Hyperbolic Tangent Restoring Force

With $R(x)$ defined as $R(x) = \tanh x$ the equation of motion becomes

$$m\ddot{x} + c_o\, f(x)\, \mathrm{sgn}\, \dot{x} + k \tanh x = 0 \quad . \qquad (4.28)$$

The equation for velocity as a function of displacement then becomes

$$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left[(-1)^n \lambda \int_x^{x_n} f(\eta)\, d\eta + \ell n\left(\frac{\cosh x_n}{\cosh x}\right)\right]} . \qquad (4.29)$$

The equation of motion and phase plane equations are given in Table 4.7 for the damping functions previously considered.

4.6 Non-odd Restoring Force

We take as a special case the non-odd restoring force $R(x) = x + \epsilon x^2$ where ϵ is a constant. The equation of motion for this case becomes

Table 4.6 Governing Equations for Restoring Force $R(x) = \sin x$

PHASE PLANE EQUATIONS FOR n^{th} HALF CYCLE
FOR MOTION SATISFYING THE EQUATION OF MOTION $m\ddot{x} + c_o f(x) \operatorname{sgn} \dot{x} + k R(x) = 0$

$f(x)$		Curves $V_o(x)$	Curves $L(x,1)$
1	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^{n+1}(x - x_n) + \cos x - \cos x_n\right)}$	A-3	A-5
$1+\alpha\lvert x\rvert$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda\left(x_n - x + \frac{\alpha}{2}(x_n^2 \operatorname{sgn} x_n - x^2 \operatorname{sgn} x)\right) + \cos x - \cos x_n\right)}$	A-3	A-6
$1+\alpha x^2$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda\left(x_n - x + \frac{\alpha}{3}(x_n^3 - x^3)\right) + \cos x - \cos x_n\right)}$	A-3	A-7 A-8
$1+\alpha\cos 4x$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda\left(x_n - x + \frac{\alpha}{4}(\sin 4x_n - \sin 4x)\right) + \cos x - \cos x_n\right)}$	A-3	A-9
$1+\alpha\lvert\tanh x\rvert$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n \lambda\left(x_2 - x + \frac{\alpha}{4}(\operatorname{sgn} x_n \,\ell n(\cosh x_n) - \operatorname{sgn} x \,\ell n(\cosh x))\right) + \cos x \cos x_n\right)}$	A-3	A-10

Table 4.7 Governing Equations for Restoring Force R(x) = tanh x

PHASE PLANE EQUATIONS FOR n^{th} HALF CYCLE
FOR MOTION SATISFYING THE EQUATION OF MOTION $m\ddot{x} + c_o\, f(x)\, \mathrm{sgn}\, \dot{x} + k\, R(x) = 0$

f(x)		Curves $V_o(x)$	Curves $L(x,1)$
1	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^{n+1}\, \lambda\, \left(x - x_n + \ell n\left(\frac{\cosh x_n}{\cosh x}\right)\right)\right)}$	A-3	A-5
$1+\alpha\lvert x\rvert$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n\, \lambda\, \left(x_n - x + \frac{\alpha}{2}\left(x_n^{\,2}\, \mathrm{sgn}\, x_n - x^2\, \mathrm{sgn}\, x\right)\right) + \ell n\left(\frac{\cosh x_n}{\cosh x}\right)\right)}$	A-3	A-6
$1+\alpha x^2$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n\, \lambda\, \left(x_n - x + \frac{\alpha}{3}\left(x_n^{\,3} - x^3\right)\right) + \ell n\left(\frac{\cosh x_n}{\cosh x}\right)\right)}$	A-3	A-7 A-8
$1+\alpha\cos 4x$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n\, \lambda\, \left(x_n - x + \frac{\alpha}{4}\,(\sin 4x_n - \sin 4x)\right) + \ell n\left(\frac{\cosh x_n}{\cosh x}\right)\right)}$	A-3	A-9
$1+\alpha\lvert\tanh x\rvert$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n\, \lambda\left(x_n - x + \alpha\left(\mathrm{sgn}\, x_n\, \ell n(\cosh x_n) - \mathrm{sgn}\, x\, \ell n(\cosh x)\right)\right) + \ell n\left(\frac{\cosh x_n}{\cosh x}\right)\right)}$	A-3	A-10

$$m\ddot{x} + c_o\, f(x)\, \operatorname{sgn} \dot{x} + k\,(x + \epsilon\, x^2) = 0 \quad . \qquad (4.30)$$

The equation describing the motion in the phase plane then becomes

$$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left[\,(-1)^n\, \lambda \int_x^{x_n} f(\eta)\, d\eta + \frac{x_n^{\,2} - x^2}{2} + \frac{\epsilon}{3}\,(x_n^{\,3} - x^3)\right]}\ . \qquad (4.31)$$

The equations listed in Table 4.8 describe the motion for five types of non-negative restoring forces.

Table 4.8 Governing Equations for Restoring Force $R(x) = x + \epsilon x^2$

PHASE PLANE EQUATIONS FOR n^{th} HALF CYCLE
FOR MOTION SATISFYING THE EQUATION OF MOTION $m\ddot{x} + c_o\, f(x)\, \mathrm{sgn}\, \dot{x} + k\, R(x) = 0$

$f(x)$		Curves $V_o(x)$	Curves $L(x,1)$
1	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left(\lambda\, (x_n - x) + \frac{x_n^2 - x^2}{2} + \epsilon\, \frac{(x_n^3 - x^3)}{3}\right)}$	A-4	A-5
$1+\alpha\lvert x\rvert$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n\, \lambda\left(x_n - x + \frac{\alpha}{2}\, (x_n^2\, \mathrm{sgn}\, x_n - x^2\, \mathrm{sgn}\, x)\right) + \frac{x_n^2 - x^2}{2} + \frac{\epsilon}{3}\left(x_n^3 - x^3\right)\right)}$	A-4	A-6
$1+\alpha x^2$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n\, \lambda\left(x_n - x + \frac{\alpha}{3}\, (x_n^3 - x^3)\right) + \frac{x_n^2 - x^2}{2} + \frac{\epsilon}{3}\left(x_n^3 - x^3\right)\right)}$	A-4	A-7 A-8
$1+\alpha\cos 4x$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n\, \lambda\left(x_n - x + \frac{\alpha}{4}\, (\sin 4x_n - \sin 4x)\right) + \frac{x_n^2 - x^2}{2} - \frac{\epsilon}{3}\left(x_n^3 - x^3\right)\right)}$	A-4	A-9
$1+\alpha\lvert\tanh x\rvert$	$\frac{y}{\omega_o} = (-1)^n \sqrt{2\left((-1)^n\, \lambda\left(x_n - x + \frac{\alpha}{4}\left(\mathrm{sgn}\, x_n\, \ell n(\cosh x_n) - \mathrm{sgn}\, x\, \ell n(\cosh x)\right)\right) + \frac{x_n^2 - x^2}{2} + \frac{\epsilon}{3}\left(x_n^3 - x^3\right)\right)}$	A-4	A-10

CHAPTER V

FREE OSCILLATIONS WITH TURBULENCE DAMPING

Damping devices which employ a gas or a liquid as the working medium often operate in the range of turbulent flow. In many of these devices the damping force is very nearly proportional to the square of the relative velocity between the damper elements. In this section we will discuss the effect of so-called "turbulence damping." We will include in our discussion the possibility that the damping force could be dependent upon displacement as well as velocity.

Let us consider the damping function introduced in equation (1.2) in the form $F_d = c_2 f(x) \dot{x}^2 \operatorname{sgn} \dot{x}$. We will assume as before that $f(x)$ is a non-negative function to avoid the effects of negative damping. The dimensions of the constant c_2 are lb-sec^2/ft^3.

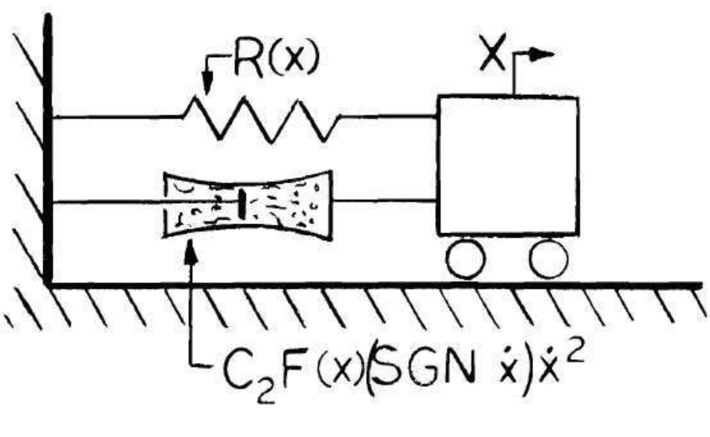


Figure 5.1 A Typical Turbulence Damped System which is Dependent Upon Displacement.

As a practical example, this type of damping might occur in a damper (which obeys the turbulent damping law) such as that shown in Figure 5.1, with a piston moving in a cylinder with variable area. Damping would be higher where the working fluid was forced through a smaller area.

The equation of motion can then be written as

$$m\ddot{x} + c_2\, f(x)\, \dot{x}^2 \operatorname{sgn} \dot{x} + k\, R(x) = 0 \quad . \tag{5.1}$$

We will retain from previous work the notation $\frac{k}{m} = \lambda$ and introduce the notation $\frac{c^2}{m} = 2\nu$ where ν is a constant whose dimensions are $1/\mathrm{ft}^2$. Using this notation and the notation

$$\ddot{x} = \dot{y} = \frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt} = y\,\frac{dy}{dx} = \frac{1}{2}\,\frac{d(y^2)}{dx}$$

we can rewrite equation (5.1) as the two first order equations $\frac{dx}{dt} = y$ and

$$\frac{d(y^2)}{dx} + 4\nu\, f(x)\, y^2 \operatorname{sgn} y = -\,2\,\omega_o^{\,2}\, R(x) \quad . \tag{5.2}$$

For convenience, we introduce the notation $x = y^2$ and note equation (5.2) is of the form

$$\frac{dz}{dx} + p(x)z = Q(x)$$

where

$$p(x) = 4\nu \operatorname{sgn} y\, f(x) \text{ and } Q(x) = -\,2\,\omega_o^{\,2}\, R(x) \quad .$$

The solution to equations of the above form can be found by use of an integrating factor or by variation of parameters.

The solution can be found as follows. From the point $x = x_1$ where $y = 0$ to the point $x = x_2$ where y is again zero, we have from (5.2) $P(x) = 4\nu \operatorname{sgn} y\, f(x) = -4\nu\, f(x)$. We can compute an integrating factor which applies for the interval from x_1 to $x = x_2$. The integrating factor is

$$e^{\int_{x_1}^{x} P(\eta)\, d\eta} = e^{-4\nu \int_{x_1}^{x} f(\eta)\, d\eta} \quad . \tag{5.3}$$

We now multiply equation (5.2) by this factor and integrate from $x = x_1$, $z = 0$ to $x = x$, $z = z$. We then have the equation

$$Z = e^{4\nu \int_{x_1}^{x} f(\eta)\, d\eta} \int_{x_1}^{x} \left(-2\, \omega_o^{2}\, R(\eta)\right) e^{-4\nu \int_{x_1}^{\eta} f(\xi)\, d\xi}\, d\eta \quad . \tag{5.4}$$

Equation (5.4) can now be rewritten as

$$\frac{y}{\omega_o} = -e^{2\nu \int_{x_1}^{x} f(\eta)\, d\eta} \sqrt{2 \int_{x}^{x_1} R(\eta)\, e^{-4\nu \int_{x_1}^{\eta} f(\xi)\, d\xi}\, d\eta} \tag{5.5}$$

which is the equation of motion in the phase plane for the first half cycle of vibration.

During the second half cycle of motion while the mass moves from x_2 to x_3 the velocity will be positive and we can write the integrating factor as

$$e^{\int_{x_2}^{x} p(\eta)\, d\eta} = e^{4\nu \int_{x_2}^{x} f(\eta)\, d\eta} \quad . \tag{5.6}$$

Multiplying equation (5.2) by this factor and integrating between the limits $x = x_2$, $z = 0$ to $x = x$, $z = z$, we have

$$Z = e^{-4\nu \int_{x_2}^{x} f(\eta)\, d\eta} \int_{x_2}^{x} 2\, \omega_o^2\, R(\eta)\, e^{4\nu \int_{x_2}^{\eta} f(\xi)\, d\xi}\, d\eta \quad . \tag{5.7}$$

The above equation which describes the motion in the phase plane for the second half cycle can be rewritten as

$$\frac{y}{\omega_o} = e^{-2\nu \int_{x_2}^{x} f(\eta)\, d\eta} \sqrt{2 \int_{x}^{x_2} R(\eta)\, e^{4\nu \int_{x_2}^{\eta} f(\xi)\, d\xi}\, d\eta} \quad . \tag{5.8}$$

As was done above, the equation of motion can be written for the n^{th} half cycle of motion where the sign of the velocity is $(-1)^n$. The equation describing motion in the phase plane becomes

$$\frac{y}{\omega_o} = (-1)^n \, e^{(-1)^{n-1} 2\nu \int_{x_n}^{x} f(\eta)\, d\eta} \sqrt{2 \int_{x}^{x_n} R(\eta)\, e^{(-1)^n 4\nu \int_{x_n}^{\eta} f(\xi)\, d\xi} \, d\eta} \; . \tag{5.9}$$

Note that $\int_x^{x_n} = \int_x^{x_o} + \int_{x_o}^{x_n} = \int_{x_o}^{x_n} - \int_{x_o}^{x}$. Thus, we can write the integral under the radical in equation (5.9) as the difference of the integrals

$$\int_{x_o}^{x_n} R(\eta)\, e^{(-1)^n 4\nu \int_{x_n}^{\eta} f(\xi)\, d\xi} \, d\eta \quad \text{and} \quad \int_{x_o}^{x} R(\eta)\, e^{(-1)^n 4\nu \int_{x_n}^{\eta} f(\xi)\, d\xi} \, d\eta \; .$$

Let us examine the second integral above

$$\int_{x_o}^{x} R(\eta)\, e^{(-1)^n 4\nu \int_{x_n}^{n} f(\xi)\, d\xi} \, d\eta = \int_{x_o}^{x} R(\eta)\, e^{(-1)^n 4\nu \left(F(\eta) - F(x_n)\right)} d\eta$$

$$= \int_{x_o}^{x} R(\eta)\, e^{(-1)^n 4\nu F(\eta)} \, e^{(-1)^{n-1} 4\nu F(x_n)} \, d\eta \tag{5.10}$$

$$= e^{(-1)^{n-1} 4\nu F(x_n)} \int_{x_o}^{x} R(\eta)\, e^{(-1)^n 4\nu F(\eta)} \, d\eta$$

We now define the function $Z(x,n)$ as

$$Z\,(x,n) = \int_{x_o}^{x} R(\eta)\, e^{(-1)^n 4\nu F(\eta)}\, d\eta \tag{5.11}$$

where $F(\eta)$ is any antiderivative of $f(\eta)$. Using the above definition we can write equation (5.10) as

$$\int_{x_o}^{x} R(\eta)\, e^{(-1)^n 4\nu \int_{x_n}^{\eta} f(\xi)\, d\xi}\, d\eta = e^{(-1)^{n-1} 4\nu F(x_n)}\, Z\,(x,n) \quad . \tag{5.12}$$

By the same method we find

$$\int_{x_o}^{x_n} R(\eta)\, e^{(-1)^n 4\nu \int_{x_n}^{\eta} f(\xi)\, d\xi}\, d\eta = e^{(-1)^{n-1} 4\nu F(x_n)}\, Z(x_n,n) \quad . \tag{5.13}$$

Having defined the function $z(x,n)$ we can now rewrite equation (5.9) as follows:

$$\frac{y}{\omega_o} = (-1)^n\, e^{(-1)^{n-1} 2\nu \int_{x_n}^{x} f(\eta)\, d\eta} \sqrt{2e^{(-1)^{n-1} 4\nu F(x_n)} \left[Z(x_n,n) - Z(x,n)\right]} \tag{5.14}$$

or

$$\frac{y}{\omega_o} = (-1)^n \, e^{(-1)^{n-1} 2\nu F(x)} \sqrt{2\left[z(x_n,n) - z(x,n)\right]} \quad . \qquad (5.15)$$

At the amplitudes x_1, x_2, x_3, ... x_N, the velocity becomes zero and since the exponential term in equation (5.16) is nonzero this requires that the radican d become zero. Thus, we can write at the point $x = x_{n+1}$

$$z(x_n,n) = z(x_{n+1},n) \quad . \qquad (5.16)$$

We can see from the definition of z(x,n) that z(x,n) = z(x,n+2) since the number n only appears as the exponent of the term $(-1)^n$ in equation (5.11).

Examples of the form of the curves z(x,1) and z(x,2) appear in Figure 5.2.

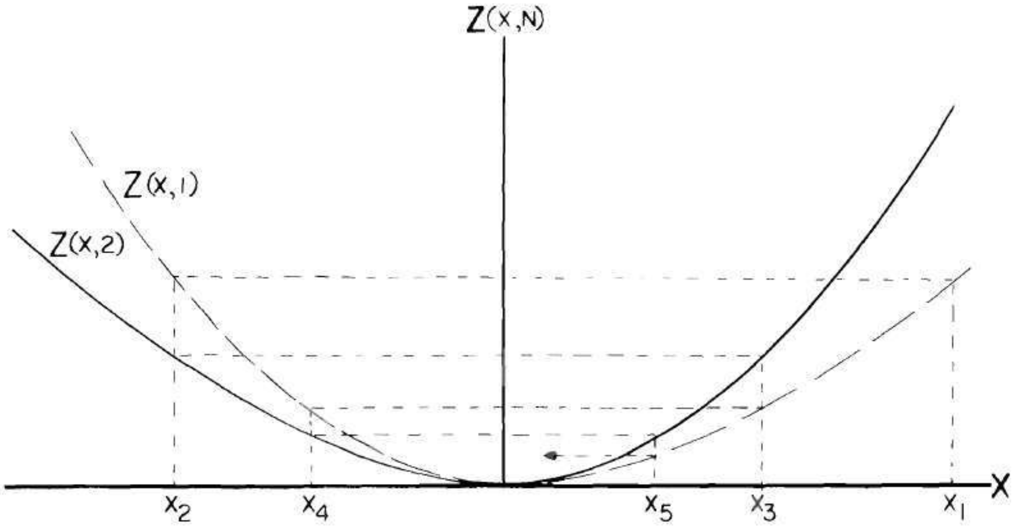


Figure 5.2 Graphical Method of Finding Successive Amplitudes of Turbulence Damped Systems.

It can be seen that the curves $z(x,1)$ and $z(x,2)$ are unsymmetrical and that they are both zero at $x = x_o^*$.

We can now find the successive amplitudes of vibration by a graphical method.

We see from equation (5.16) that if we drew a horizontal line from $z(x,1)$ that it would intersect the curve $z(x,1)$ at the point $z(x_2,1)$. Thus, the unknown amplitude x_2 has been found. From equation (5.16) we can write $z(x_2,2) = z(x_3,2)$; thus, if we drew a horizontal line from $z(x_2,2)$ we could find the unknown amplitude x_3. Similarly, $z(x_3,3) = z(x_4,3)$ and from (5.18) we see $z(x_3,3) = z(x_3,1)$ and $z(x_4,3) = z(x_4,1)$. Thus, the location of amplitude x_4 can be found in the same way x_2 was found by drawing a horizontal line from the point $z(x_3,1)$ to $z(x_4,1)$. The method continues as described above and in Figure 5.2 with each amplitude being smaller than the one preceding it.

Notice that the rate of decay of amplitudes is dependent upon the magnitude of the damping constant ν. The curves $z(x,n)$ are symmetrical and superimposed upon each other for a value $\nu = 0$; thus, the graphical method gives no decrease in vibration amplitude as would be expected. For increasing values of ν the curves become more unsymmetrical resulting in a more rapid decay of successive amplitudes.

The time required for the n^{th} half cycle of vibration could be found by integrating equation (5.10). The resulting equation appears below.

---

*As before, we will assume the equilibrium position is $x_o = 0$.

$$t_n = \int_{x_n}^{x_{n+1}} \frac{dx}{y}$$

$$= \int_{x_n}^{x_{n+1}} \frac{dx}{(-1)^n e^{(-1)^{n-1} 2\nu \int_{x_n}^{x} f(\eta)\, d\eta} \sqrt{2 \int_{x}^{x_n} R(\eta)\, e^{(-1)^n 4\nu \int_{x_n}^{\eta} f(\xi)\, d\xi}\, d\eta}}\,.$$

In general, equation (5.17) will be a complicated equation requiring numerical integration.

In the following two chapters the equations of motion and the equations of motion in the phase plane will be given for various restoring forces and damping functions. In Chapter VI it will be assumed that $f(x) = 1$, and in Chapter VII the results for other non-negative functions will be introduced.

CHAPTER VI

CONSTANT TURBULENCE DAMPING

In this chapter we will assume the damping function $f(x) = 1$. In this case equation (5.9) which describes the motion in the phase plane during the n^{th} half cycle of vibration can be written

$$\frac{y}{\omega_o} = (-1)^n e^{(-1)^{n-1} 2\nu (x-x_n)} \sqrt{2 \int_x^{x_n} R(\eta)\, e^{(-1)^n 4\nu (\eta - x_n)}\, d\eta}\ , \qquad (6.1)$$

or

$$\frac{y}{\omega_o} = (-1)^n e^{(-1)^{n-1} 2\nu x} \sqrt{2} \sqrt{\int_x^{x_n} R(\eta)\, e^{(-1)^n 4\nu \eta}\, d\eta}\ . \qquad (6.2)$$

For convenience, we will define

$$A_n(x) = (-1)^n e^{(-1)^{n-1} 2\nu x} \sqrt{2}\ , \qquad (6.3)$$

which allows us to condense equation (6.2) to the more concise form

$$\frac{y}{\omega_o} = A_n(x) \sqrt{\int_x^{x_n} R(\eta)\, e^{(-1)^n 4\nu \eta}\, d\eta}\ . \qquad (6.4)$$

6.1 Linear Restoring Force

Some vibration problems found in practice follow (or closely approximate) the law of turbulence damping and linear restoring force.

These problems follow the equation of motion

$$m\ddot{x} + c_2\, \dot{x}^2 \operatorname{sgn} \dot{x} + k\, x = 0 \quad . \tag{6.5}$$

We can then write from equation (5.10) the expression for velocity as a function of displacement for the n^{th} half cycle. It is

$$\frac{y}{\omega_o} = A_n(x)\sqrt{(-1)^n \left[e^{(-1)^n\, 4\nu\, x_n}\left(4\nu\, x_n - (-1)^n\right) - e^{(-1)^n\, 4\nu\, x}\left(4\nu\, x - (1)^n\right)\right]} \tag{6.6}$$

Figure A-11 gives curves for use in determining successive amplitudes for several damping ratios. Notice that for higher values of damping the motion dies out more rapidly and all motion takes an infinite time to die out.

6.2 Parabolic Restoring Force

As a special case we take the restoring force $R(x) = x^3$. Thus, the equation of motion becomes

$$m\ddot{x} + c_2\, \dot{x}^2 \operatorname{sgn} \dot{x} + k\, x^3 = 0 \quad . \tag{6.7}$$

The phase plane equation for the n^{th} half cycle is

$$\frac{y}{\omega_o} = \frac{A_n(2)}{16\nu^2}\left\{e^{(-1)^n\, 4\nu\, x_n}\left[(-1)^n\, 64\nu^3\, x_n{}^3 - 48\nu^2\, x_n{}^2 + (-1)^n\, 24\nu\, x_n - 6\right) \right. \tag{6.8}$$

$$\left. - e^{(-1)^n\, 4\nu\, x}\ (-1)^n\, 64\nu^3\, x^3 - 48\nu^2\, x^2 + (-1)^n\, 24\nu\, x - 6 \right]\Bigg\}^{\frac{1}{2}} \quad .$$

Figure A-12 contains curves for graphical solutions.

6.3 Duffing Type Restoring Force

With $R(x) = x + \epsilon x^3$ the equation of motions reads

$$m\ddot{x} + c_2 \dot{x}^2 \operatorname{sgn} \dot{x} + k (x + \epsilon x^3) = 0 \quad ; \tag{6.9}$$

thus, a closed expression for velocity as a function of displacement for the n^{th} half cycle is

$$\begin{aligned} \frac{y}{\omega_o} = A_n(x) \Bigg\{ & \frac{(-1)^n}{16\nu^2} \Big[e^{(-1)^n 4\nu x_n} \Big(4\nu x_n - (-1)^n\Big) - e^{(-1)^n 4\nu} \Big(4\nu x - (-1)^n\Big) \Big] \\ & + \frac{\epsilon}{256\nu^4} \Big[e^{(-1)^n 4\nu x_n} \Big((-1)^n 64\nu^3 x_n^{\ 3} - 48\nu^2 x_n^{\ 2} + (-1)^n 24\nu x_n - 6\Big) \\ & - e^{(-1)^n 4\nu x} \Big((-1)^n 64\nu^3 x^3 - 48\nu^2 x^2 + (-1)^n 24\nu x - 6\Big) \Big] \Bigg\}^{\frac{1}{2}} . \end{aligned} \tag{6.10}$$

Curves are given in Figure A-13 for $\epsilon = 0.01$.

6.4 Sinusoidal Restoring Force

The equation of motion is

$$m\ddot{x} + c_2 \dot{x}^2 \operatorname{sgn} \dot{x} + k \sin x = 0 \quad . \tag{6.11}$$

For the n^{th} half cycle the phase plane equation reads

$$\begin{aligned} \frac{y}{\omega_o} = A_n(x) \Bigg\{ & \frac{1}{16\nu^{2+1}} \Big(e^{(-1)^n 4\nu x_n} \Big[(-1)^n 4\nu \sin x_n - \cos x_n \Big] \\ & - e^{(-1)^n 4\nu x} \Big[(-1)^n 4\nu \sin x - \cos x \Big] \Big) \Bigg\}^{\frac{1}{2}} . \end{aligned} \tag{6.12}$$

See Figure A-14 for graphical solutions.

6.5 Hyperbolic Tangent Restoring Force

The equation of motion is

$$m\ddot{x} + c_2 \dot{x}^2 \operatorname{sgn} \dot{x} + k \tanh x = 0 \quad . \tag{6.13}$$

To obtain the phase plane solution we note that for $x^2 < \frac{\pi^2}{4}$, $\tanh x$ can be written as an infinite series

$$\tanh x = x - \frac{x^3}{3} + \frac{2}{15} x^5 - \frac{17}{315} x^7 + \frac{62}{2835} x^9 \ . \ . \ . \ .$$

$$. \ . \ . \ . + \frac{(-1)^{n-1} \, 2^{2n} \, (2^{2n} - 1) \, B_n \, x^{2n-1}}{(2n)!} + . \ . \ . \ . \quad , \tag{6.14}$$

where B_n is the n^{th} Bernoulli number, and is given by

$$B_n = \frac{(2n)!}{\pi^{2n} \, 2^{2n-1}} \left[1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + . \ . \ . \ . \right] .$$

As an approximation to tanh x we take the first two terms of the above series. Thus, an approximate phase plane equation for the n^{th} half cycle is

$$\frac{y}{\omega_o} = A_n(2) \left\{ \frac{1}{16\nu^2} \left[(-1)^n e^{(-1)^n 4\nu x_n} \left(4\nu x_n - (-1)^n \right) - (-1)^n e^{(-1)^n 4\nu x} \left(4\nu x - (-1)^n \right) \right] - \frac{1}{3} \left[\frac{1}{256\nu^4} \left\{ \left(e^{(-1)^n 4\nu x_n} \left[(-1)^n 64\nu^3 x_n^{\,3} - 48\nu^2 x_n^{\,2} + (-1)^n 24\nu x_3 - 6 \right] \right) - e^{(-1)^n 4\nu x} \left((-1)^n 64\nu^3 x^3 - 48\nu^2 x^2 + (-1)^n 24\nu x_n - 6 \right) \right\} \right] \right\}^{\frac{1}{2}} . \tag{6.15}$$

The above approximation to tanh x yields a vibrator of the Duffing type for $\epsilon = -\frac{1}{3}$.

If more accuracy is desired in the approximation more terms can be added to the approximation of tanh x.

Curves for a hyperbolic tangent restoring force are given in Figure A-15 for graphical solution.

6.6 Non-odd Restoring Force

With $R(x) = x + \epsilon x^2$ the equation of motion is

$$m\ddot{x} + c_2 \dot{x}^2 \operatorname{sgn} \dot{x} + k (x + \epsilon x^2) = 0 \quad . \tag{6.16}$$

Thus, the phase plane equation which applies during the n^{th} half cycle can be written as

$$\frac{y}{\omega_o} = A_n(2)\left\{\frac{1}{16\nu^2}\left((-1)^n\left[e^{(-1)^n 4\nu x_n}\left(4\nu x_n - (-1)^n\right) - e^{(-1)^n}\left(4\nu x - (-1)^n\right)\right]\right) + \epsilon\left[\frac{(-1)^n e^{(-1)^n 4\nu x_n}}{64\nu^3}\left(16\nu^2 x_n^{\ 2} - (-1)\, 8\nu x_n + 2\right) + \frac{(-1)^n e^{(-1)^n 4\nu x}}{64\nu^3}\left(16\nu^2 x^2 - (-1)^n 8\nu x + 2\right)\right]\right\}^{\frac{1}{2}} . \tag{6.17}$$

Curves appear in Figure A-16 for $\epsilon = 0.1$.

CHAPTER VII

DISPLACEMENT DEPENDENT TURBULENCE DAMPING

If the turbulence damping device is affected by its position as in Figure 5.1 the equation of motion could then be written in the form (5.1)

$$m\ddot{x} + c_2 \; f(x) \; \dot{x}^2 \; \mathrm{sgn} \; \dot{x} + k \; R(x) = 0 \quad . \tag{7.1}$$

The procedure outlined in Chapter V gives a general expression for the velocity during the n^{th} half cycle. This expression, equation (5.10), is rewritten for convenience below.

$$\frac{y}{\omega_o} = (-1)^n \; e^{(-1)^{n-1} \; 2\nu \int_{x_n}^{x} f(\eta) \; d\eta} \sqrt{2 \int_{x}^{x_n} R(\eta) \; e^{(-1)^n \; 4\nu \int_{x_n}^{\eta} f(\xi) \; d\xi} \; d\eta} \; . \tag{7.2}$$

It can be seen that for any but the most simple functions, such as $f(x) = 1$, the above equation can be solved only by numerical methods. Since in these cases closed solutions cannot be given, the results will be given in the form of curves. The results for functions $f(x) = 1 + \alpha \; x^2$ and $f(x) = 1 + \alpha \cos 4x$ for $0 \le \alpha \le 1$ are shown in Figures A.17 through A.22. These curves of $Z(x,n)$ can be used to find successive amplitudes for any given displacement.

CHAPTER VIII

FREE OSCILLATIONS WITH A COMBINATION OF COULOMB AND TURBULENCE DAMPING

In physical systems it is often difficult if not impossible to obtain damping of one type only. In viscous dampers of the turbulence type one might encounter coulomb friction in the damper itself, or in the case of a pendulum it is likely that friction will be present to some degree in the pivot. In such cases it may be more accurate to assume both types of damping are present. In the following it will be assumed that the equation of motion can be written in the following form

$$m\ddot{x} + \left(c_o\ f(x) + c_2\ \dot{x}^2\ g(x)\right) \operatorname{sgn} \dot{x} + k\ R(x) = 0 \quad . \qquad (8.1)$$

The constants c_o and c_2 have been defined previously and $f(x)$ and $g(x)$ are non-negative functions. Note that the proportions of each type of damping can be adjusted to fit different systems. We will retain the notation

$$\frac{k}{m} = \omega_o^{\ 2} \ , \ \frac{c_o}{m} = \omega_o^{\ 2}\ \lambda \ , \text{ and } \frac{c_2}{m} = 2\nu \quad .$$

Equation (8.1) can be written as

$$y\,\frac{dy}{dx} + 2\nu \operatorname{sgn} yg\ (x)\ y^2 = -\ \omega_o^{\ 2}\left(R(x) - \lambda \operatorname{sgn} y\ f(x)\right) . \qquad (8.2)$$

Or using the notation $z = y^2$,

$$\frac{dz}{dx} + 4\nu \operatorname{sgn} y\, g(x)\, z = -2\,\omega_o^{\,2}\Big(R(x) + \lambda \operatorname{sgn} y\, f(x)\Big) \quad . \tag{8.3}$$

The solution of equation (8.3) can be found by using an integrating factor as was done in Chapter V since it is of the form

$$\frac{dz}{dx} + P(x)\, z = Q(x) \quad , \tag{8.4}$$

where

$$P(x) = 4\nu \operatorname{sgn} y\, g(x)$$

$$Q(x) = -2\,\omega_o^{\,2}\Big(R(x) + \operatorname{sgn} y\, f(x)\Big) \quad .$$

With the above notation the expression for $y(x)$ can be found by a similar method to that used in Chapter V. We have from equation (5.10) only to change the terms corresponding to $Q(x)$ to obtain the equation of the motion in the phase plane for the n^{th} half cycle of motion.

For the n^{th} half cycle of motion we can write (assuming $x_1 > x_o$)

$$\frac{y}{\omega_o} = (-1)^n \sqrt{2}\; e^{(-1)^{n-1}\, 2\nu \int_{x_n}^{x} g(\eta)\, d\eta} \sqrt{\int_x^{x_n}\Big(R(\eta) + (-1)^n \lambda\, f(\eta)\Big)\, e^{(-1)^n\, 4\nu \int_{x_n}^{\eta} g(\xi)\, d\xi}\, d\eta} \quad . \tag{8.5}$$

In any but the most simple cases, i.e., $g(x) = 1$, the above equation is complicated and the solution can be most easily obtained by numerical integration. The solution of equation (8.5) is beyond the scope of this presentation, however, and is not included.

CHAPTER IX

RESULTS AND CONCLUSIONS

The phase plane method has been applied to several single degree of freedom vibration systems with varied complexity for "coulomb" damped systems, "turbulence" damped systems, and a combination of "coulomb" and "turbulence" damped systems.

For "coulomb" damped systems the graphical solution for consecutive amplitudes involved drawing a curve of $V_o(x) = \int_{x_o}^{x} R(\eta)\, d\eta$ for each type of restoring force $R(x)$. Then an overlay was prepared according to the equation $L(x,n) = (-1)^n \lambda \int_{x_o}^{x} f(\eta)\, d\eta$ for each non-negative damping function $f(x)$ and each value of damping rate λ. It was shown that the consecutive amplitudes could be obtained by superposition of the curves $V_o(x)$ and $L(x,n)$. The above curves are given in the appendix for use in finding consecutive amplitudes for all of the combinations of restoring forces and damping forces considered in the text. It was found that "coulomb" damped vibration ceased at a finite amplitude for which $R(x) \geq \lambda$ except in one instance when $R(x) = \operatorname{sgn} x$, and $f(x) = 1$. In this case vibration once begun continued indefinitely with the $(n+1)^{st}$ amplitude being given by $x_{n+1} = \left[- \frac{(1-\lambda)}{1+\lambda} \right]^n x_1$.

In the case of "turbulence" damped vibrations the consecutive amplitudes could be found by plotting two curves defined by the equation

$$Z(x,n) = \int_{x_o}^{x} R(n)\ e^{(-1)^n\ 4\nu\ F(\eta)}\ d\eta \quad .$$

It was shown that once the above curves were drawn for a given damping ν, and functions $R(x)$ and $f(x)$, that one could begin with a given initial amplitude x_1 and by superimposing a system of horizontal lines upon the curves for $Z(x,n)$ could locate each consecutive amplitude. It was found that the vibration did not cease at a finite amplitude as was the case with the "coulomb" vibrators but that motion continued to die out and reached the point of equilibrium in an infinite time.

In those cases where the damping force was a non-negative function of displacement the consecutive amplitudes were found to die down more quickly. In some cases, notably the "Duffing" vibrator with $\epsilon < 0$, initial displacements were limited if vibration was to take place about the equilibrium point $x_o = 0$. For vibrators of the non-odd type initial amplitudes were also limited.

The governing equations were written and curves were drawn for many specific cases, but the methods outlined are given in general form and can be used for any combination of restoring force and non-negative damping force.

APPENDIX

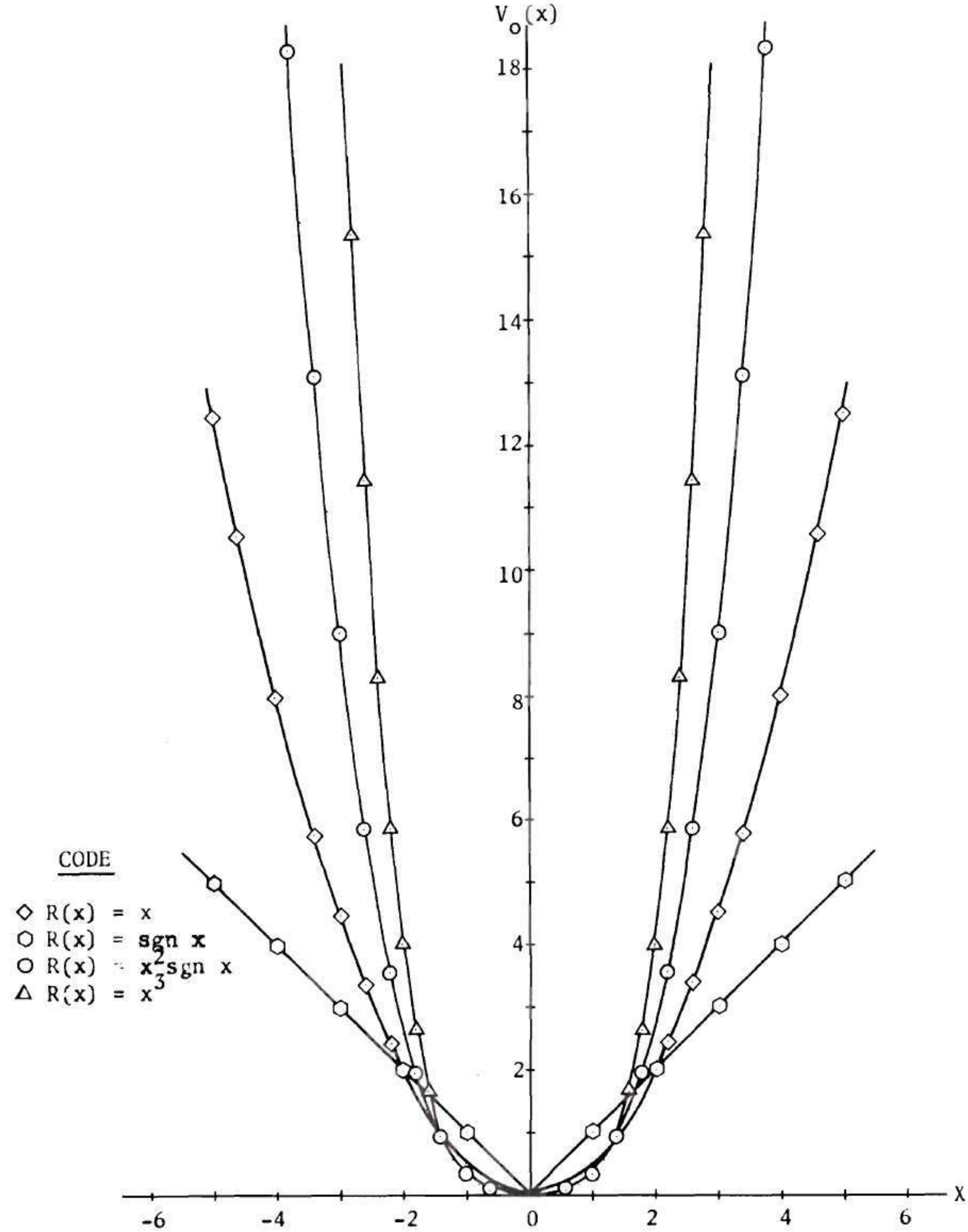


Figure A-1: Curves of $V_o(x)$ for Parabolic Restoring Forces.

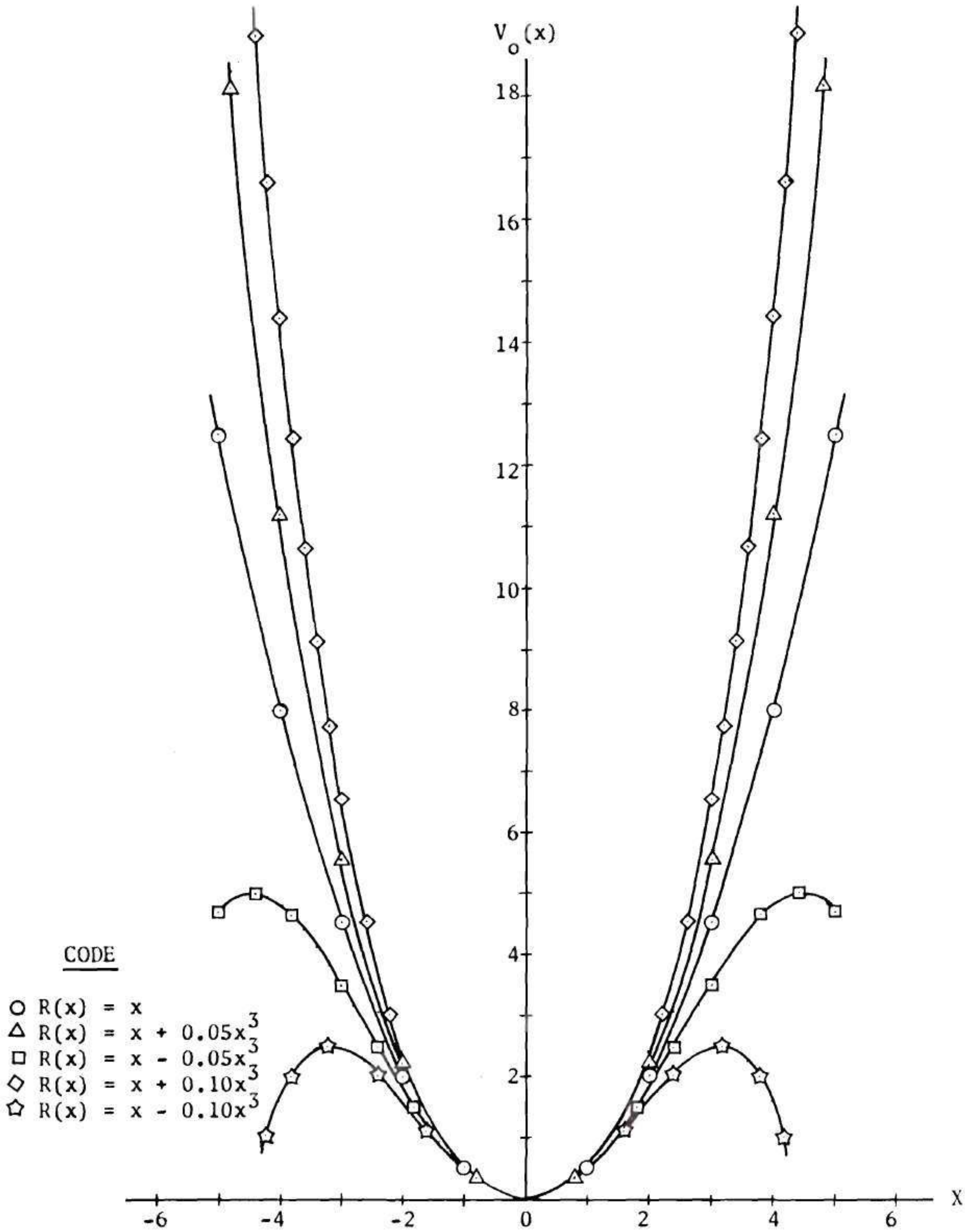


Figure A-2: Curves of $V_o(x)$ for Duffing Type Restoring Forces.

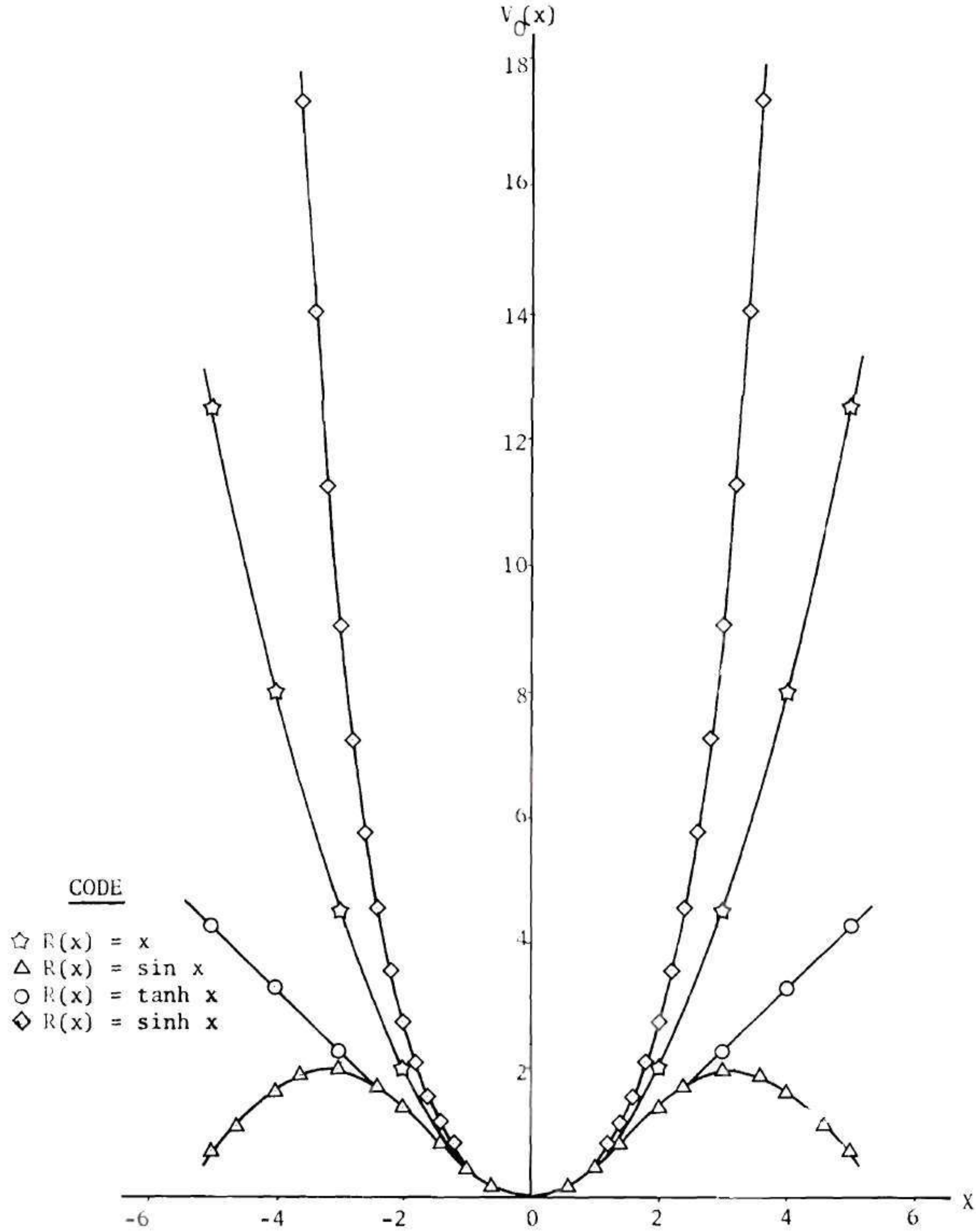


Figure A-3: Curves of $V_o(x)$ for Trigonometric Type Restoring Forces.

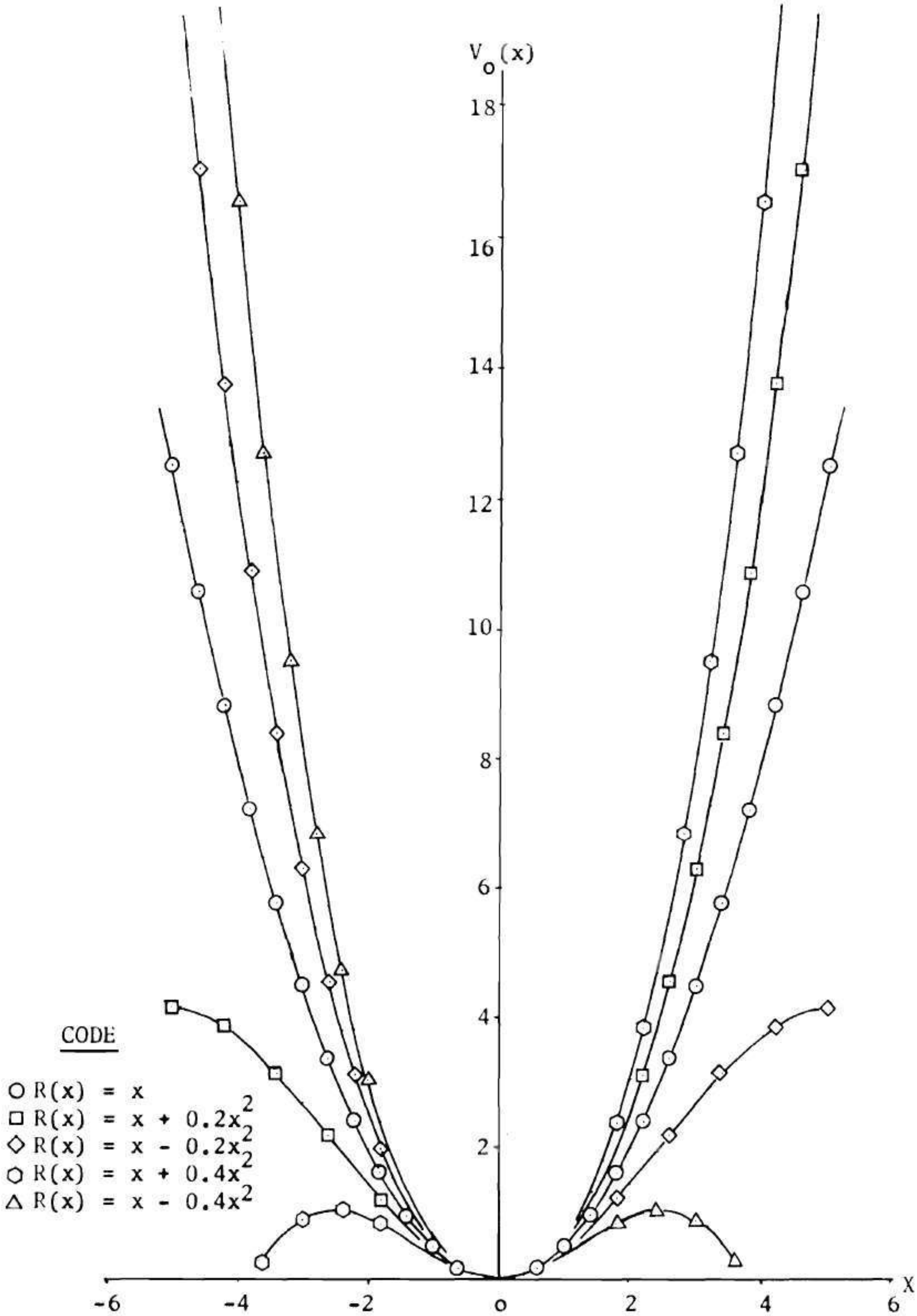


Figure A-4: Curves of $V_o(x)$ for Non Odd Restoring Forces.

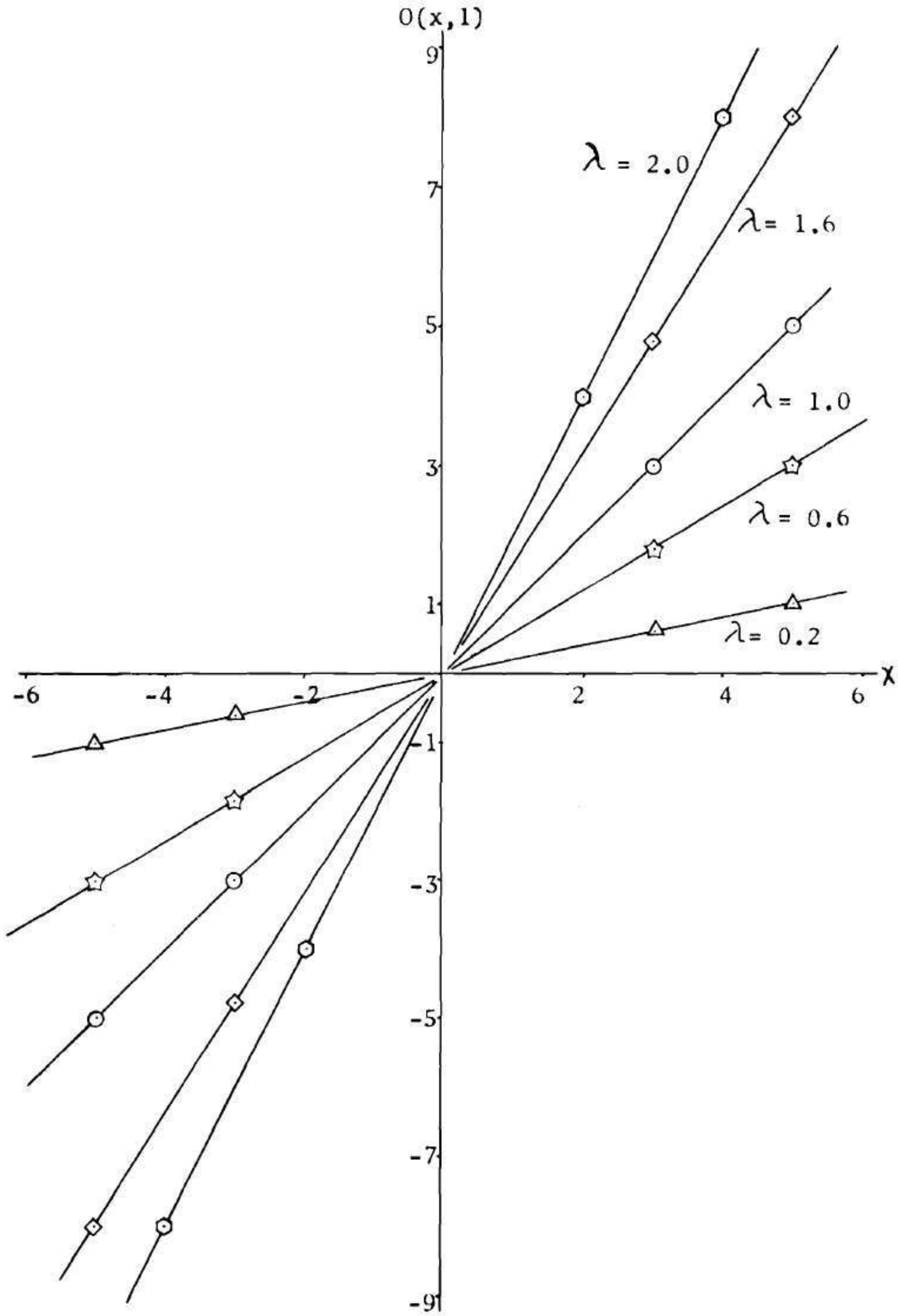


Figure A-5: Overlay Curves for Damping Function $f(x) = 1$ ($\lambda = c_o/k$).

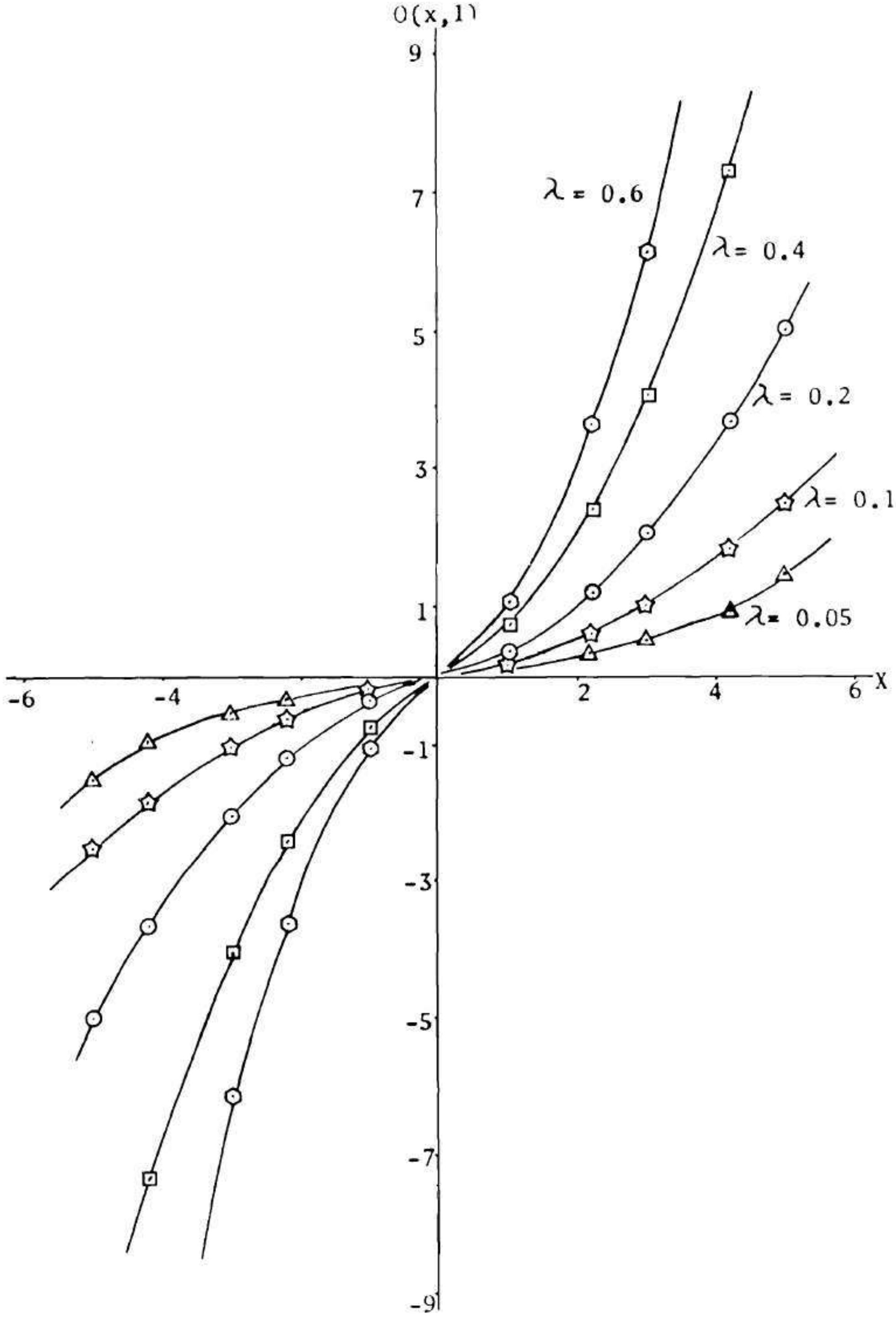


Figure A-6: Overlay Curves for [illegible]ing Function $f(x) = 1 + 0.4|x|$ ($\lambda = c_0/k$).

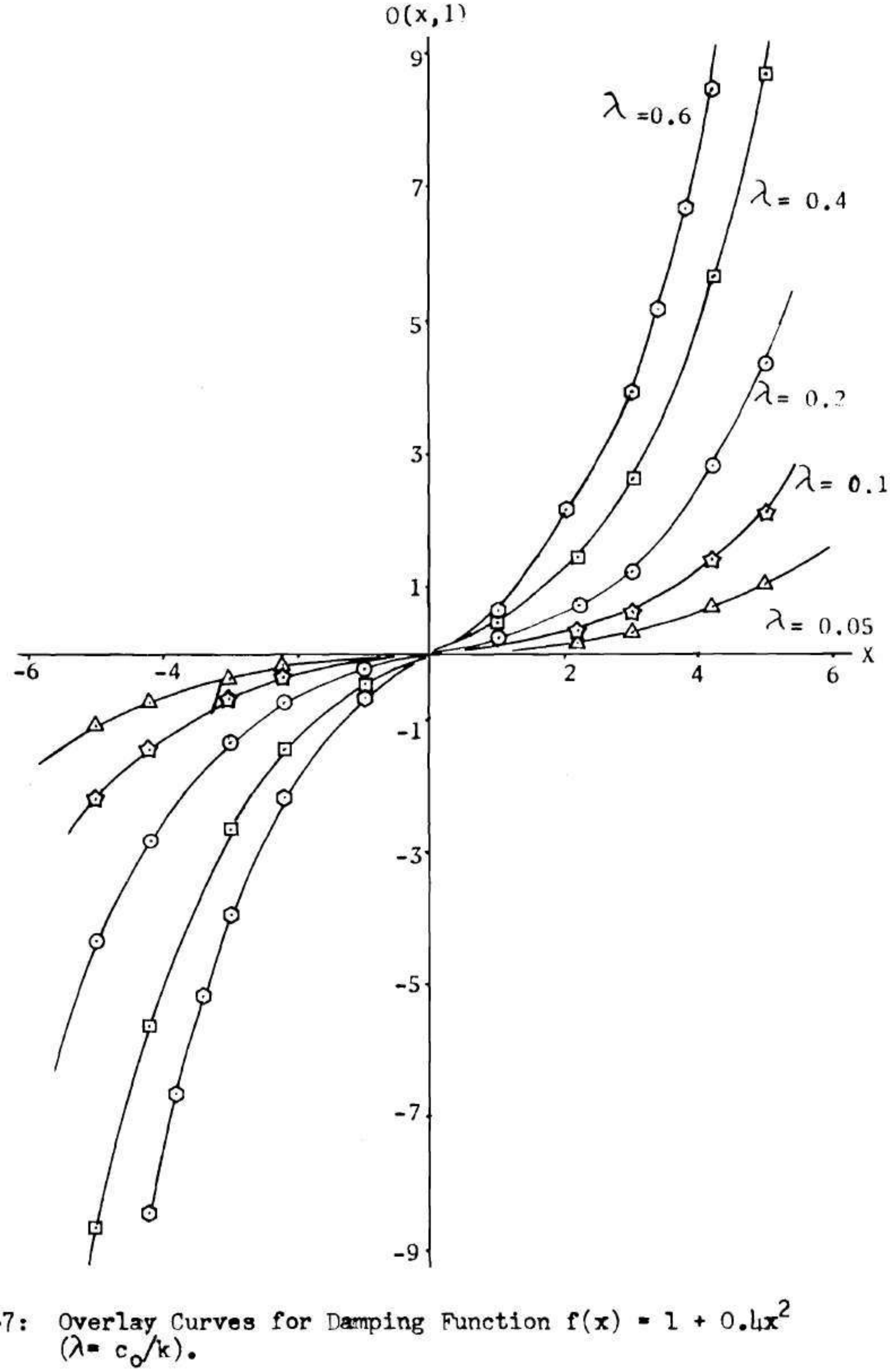


Figure A-7: Overlay Curves for Damping Function $f(x) = 1 + 0.4x^2$ ($\lambda = c_o/k$).

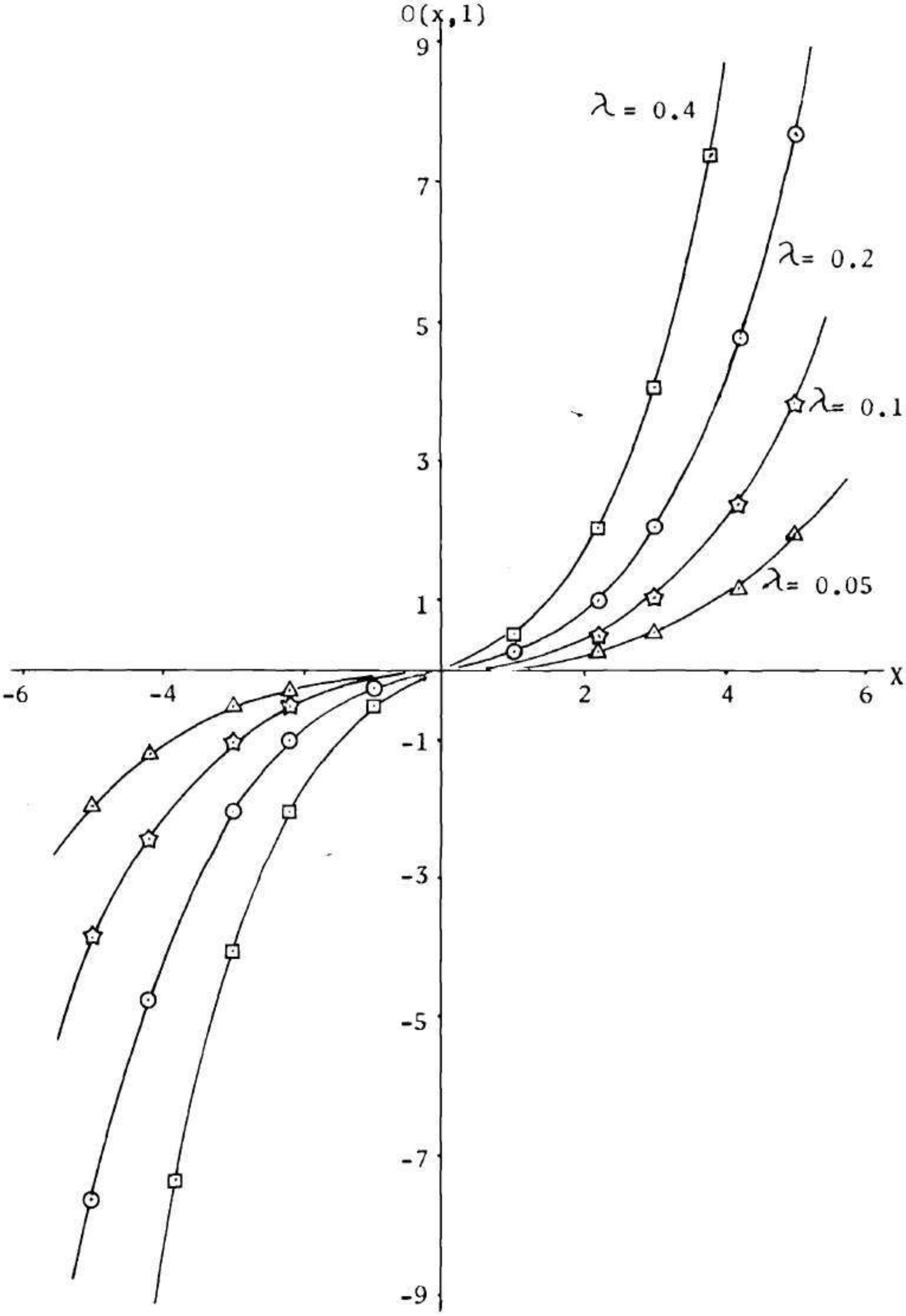


Figure A-8; Overlay Curves for Damping Function $f(x) = 1 + 0.8x^2$ ($\lambda = c_0/k$).

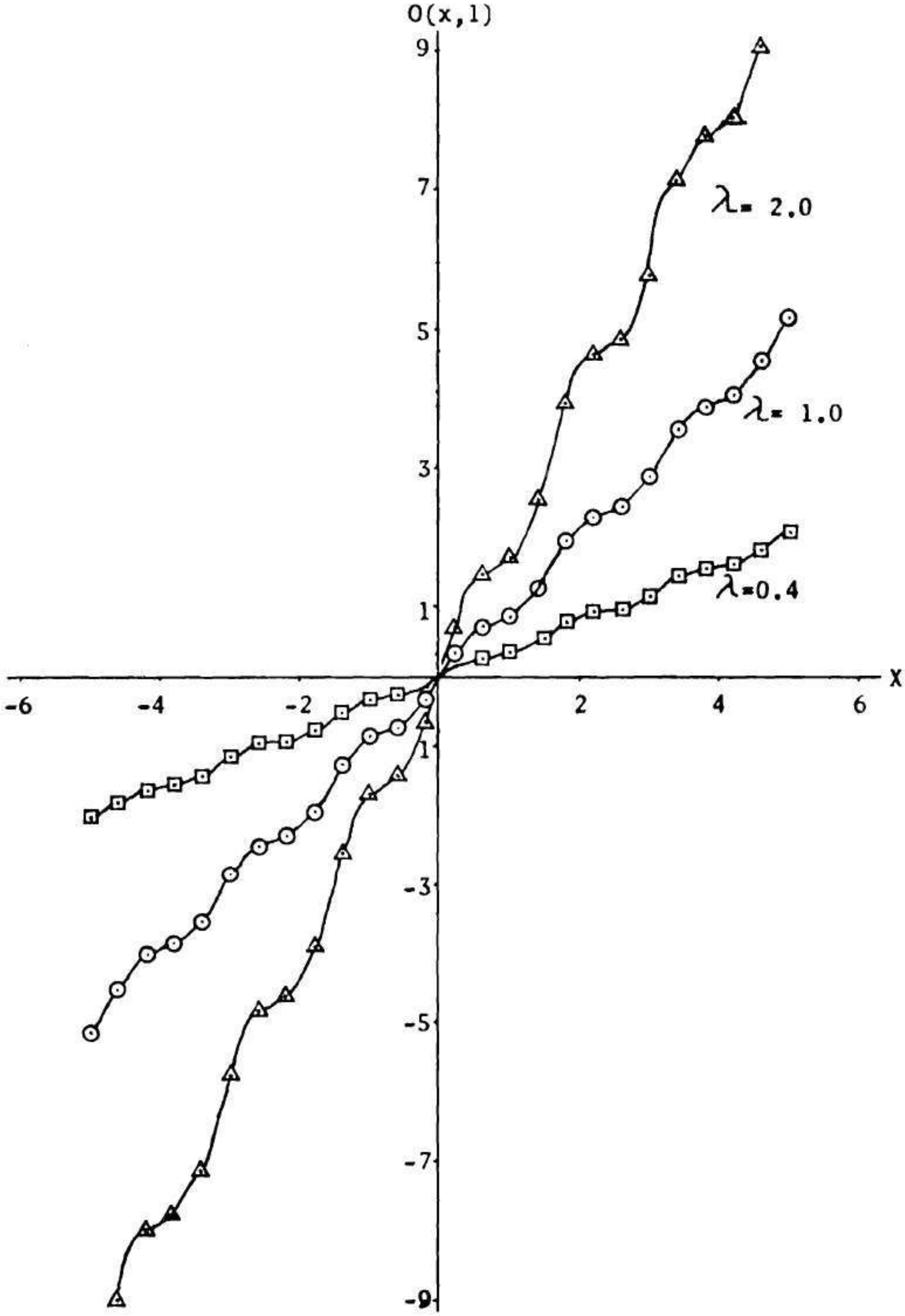


Figure A-9: Overlay Curves for Damping Function $f(x) = 1 + \cos \lambda x$ ($\lambda = c_o/k$)

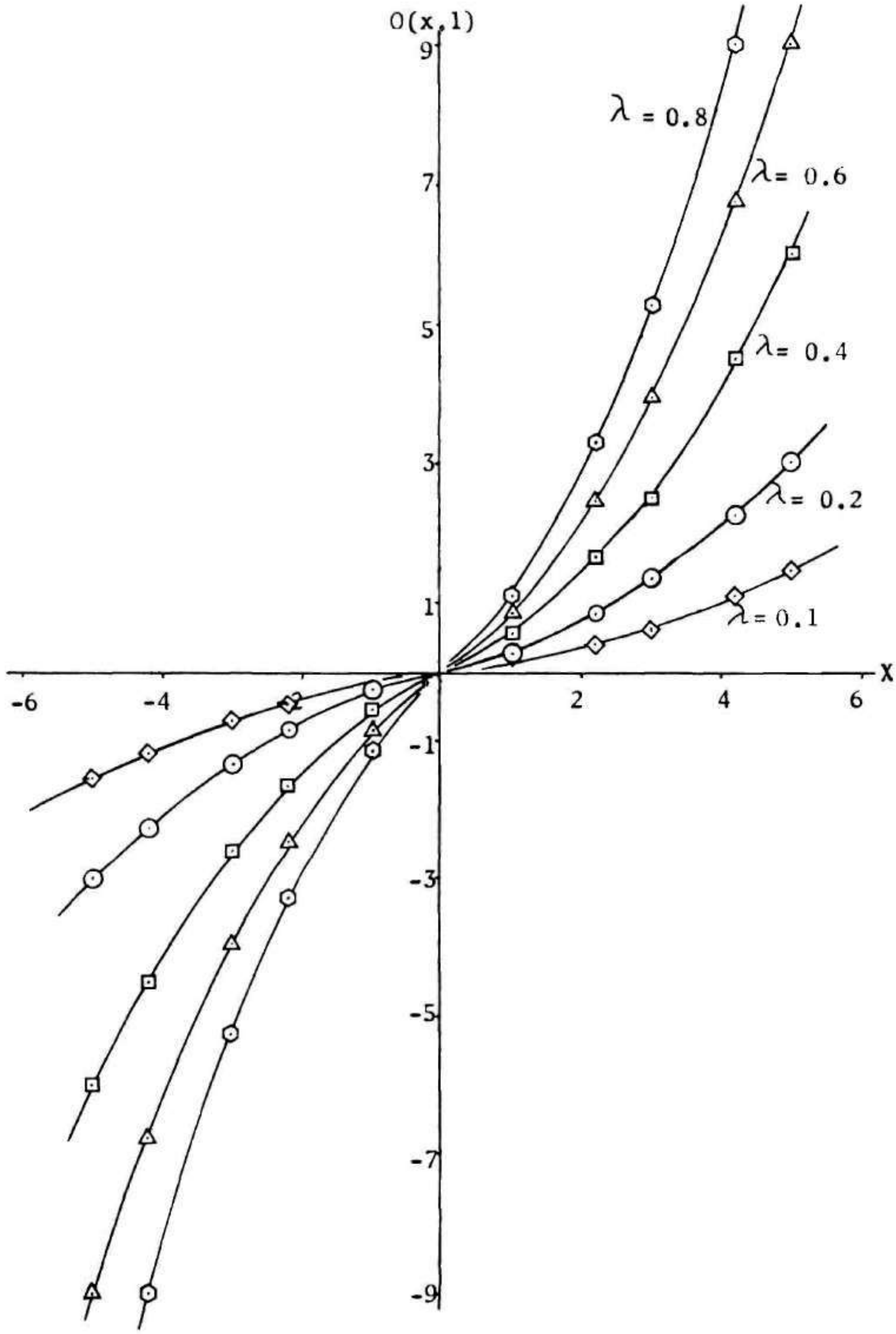


Figure A-10: Overlay Curves for Damping Function $f(x) = 1 + 0.8\,|\tanh x|$
($\lambda = c_o/k$)

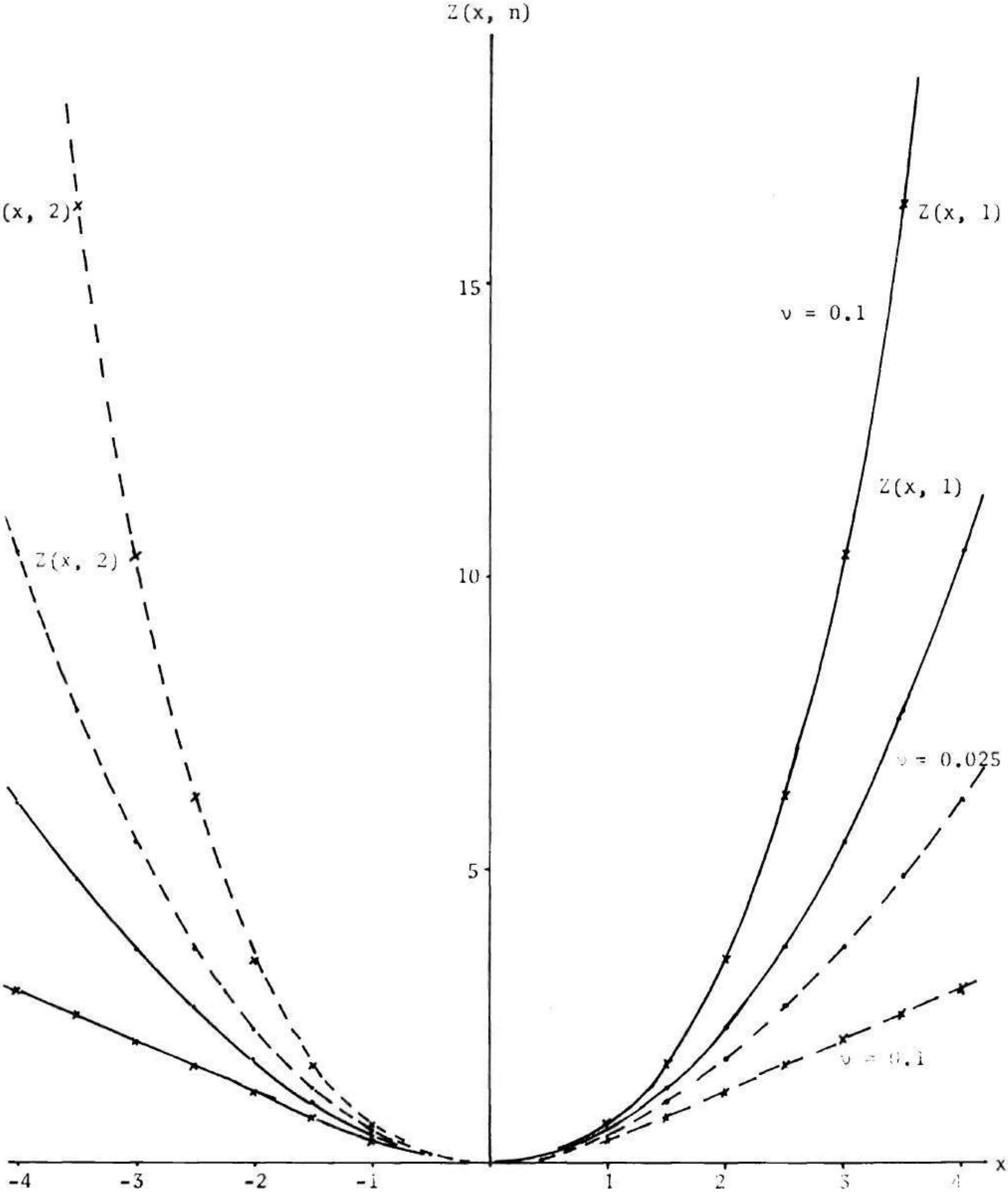


Figure A-11: $Z(x, n)$ curves for $P(x) = x$, $f(x) = 1$, ($\nu = c_2/2m$)

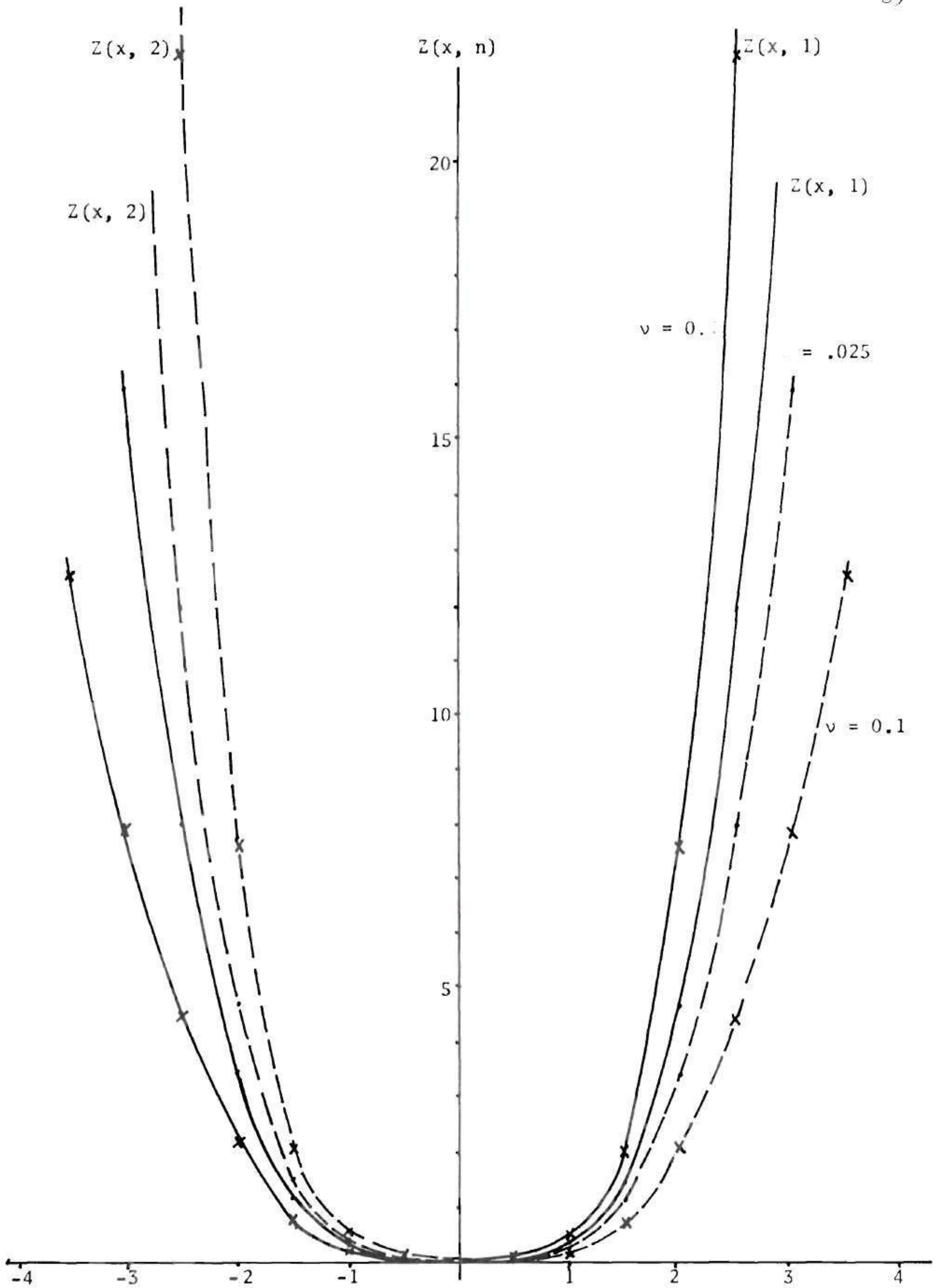


Figure A-12: $Z(x, n)$ curves for $R(x) = x^3$, $f(x) = 1$, ($\nu = c_2/2r$).

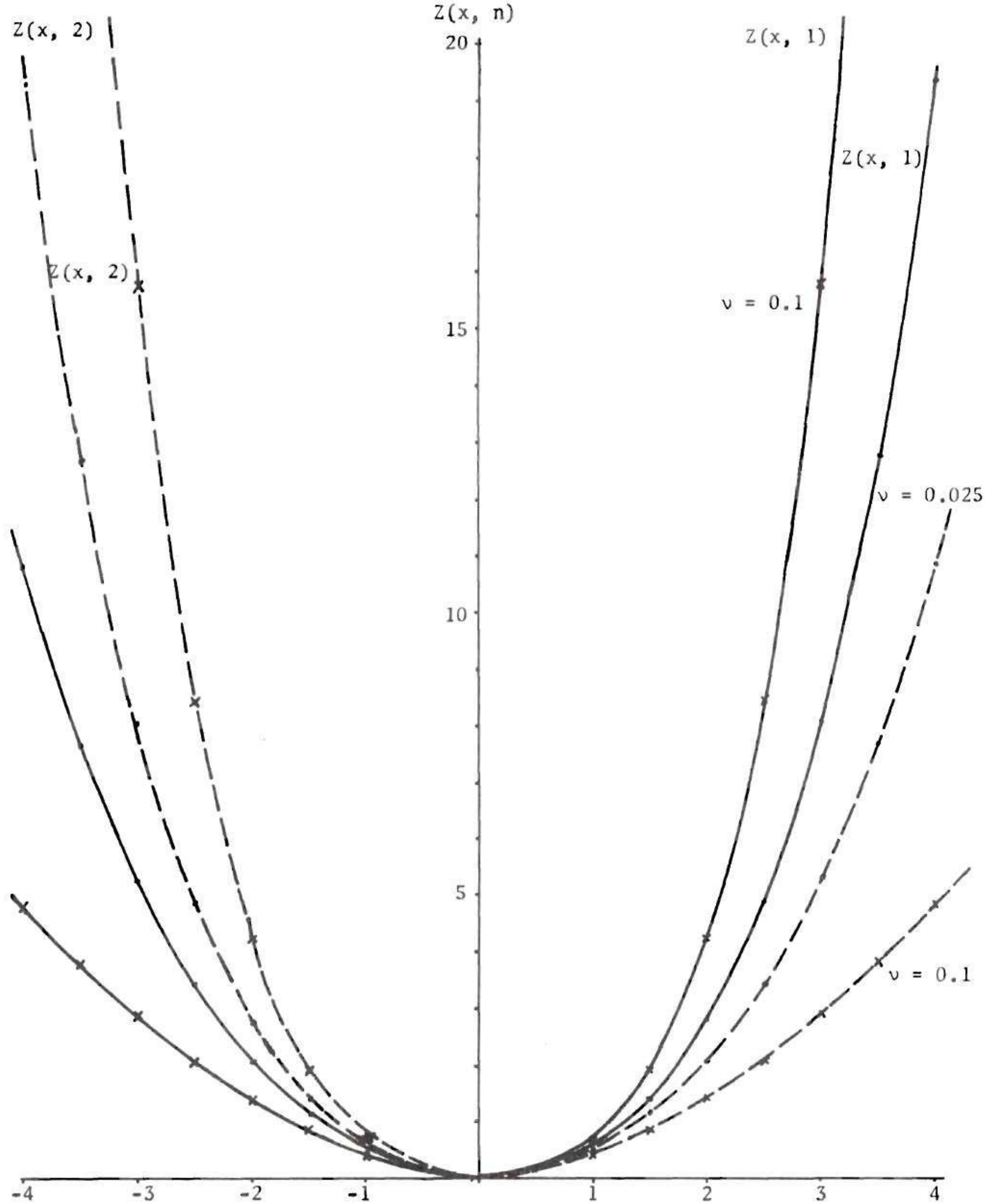


Figure A-13: $Z(x,\ n)$ curves for $R(x) = x + \epsilon x^3$, $f(x) = 1$, ($\nu = c_2/2m$).

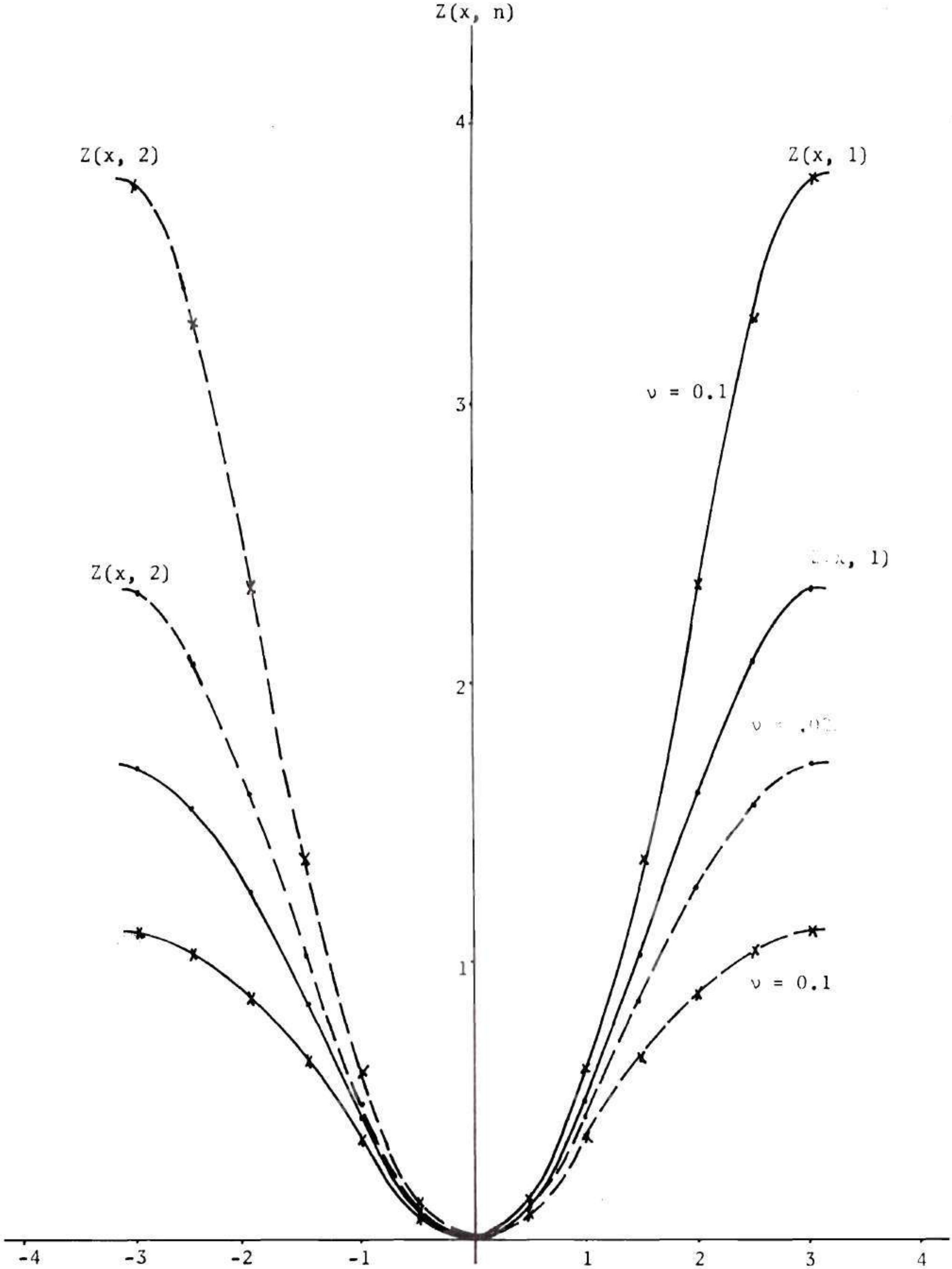


Figure A-14: Z(x, n) curves for R(x) = sin x, f(x) = 1, ($\nu = c_2/2m$).

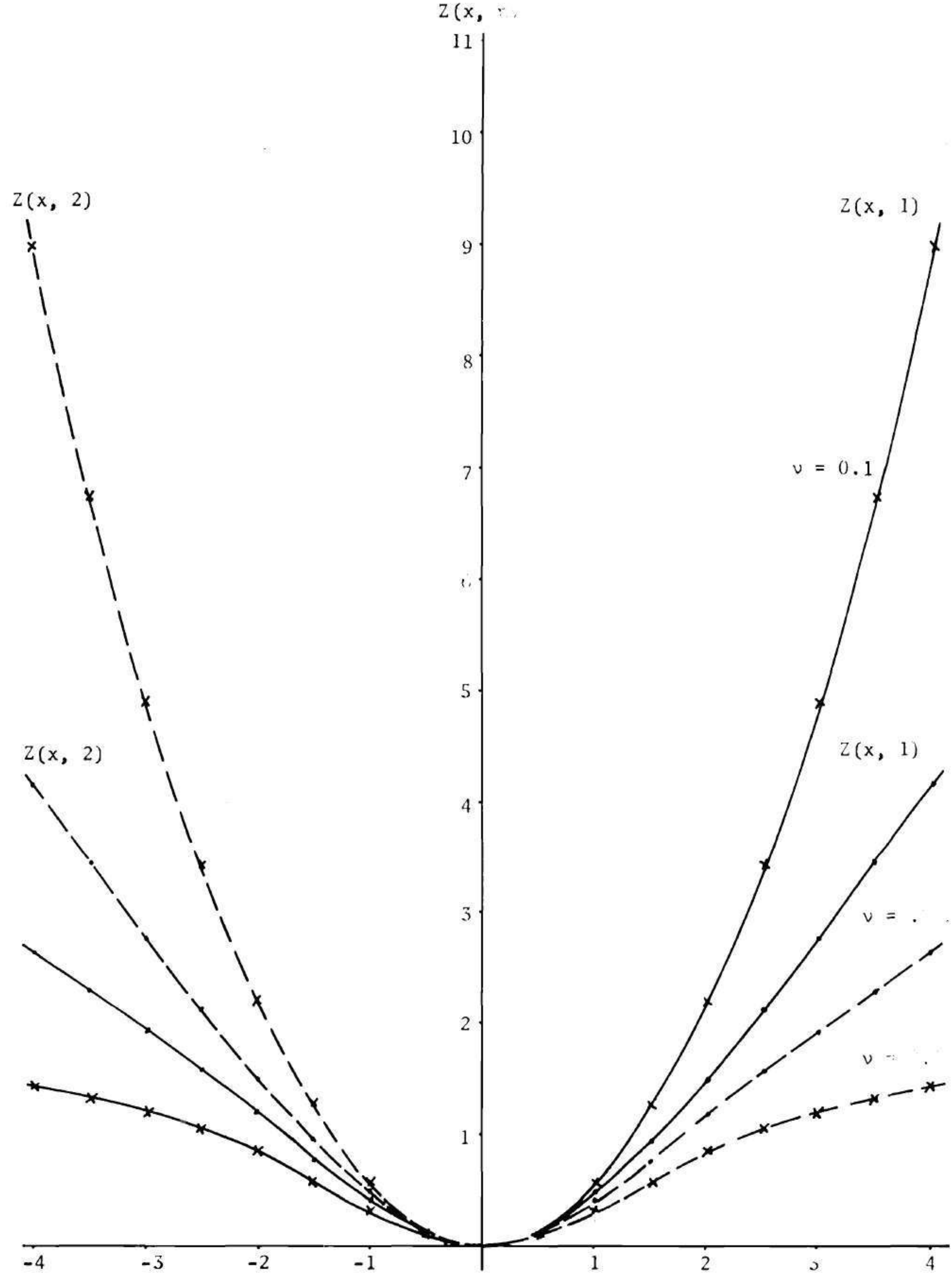


Figure A-15: Z(x, n) curves for R(x) = tanh x, f(x) = 1, ($\nu = c_2/2m$).

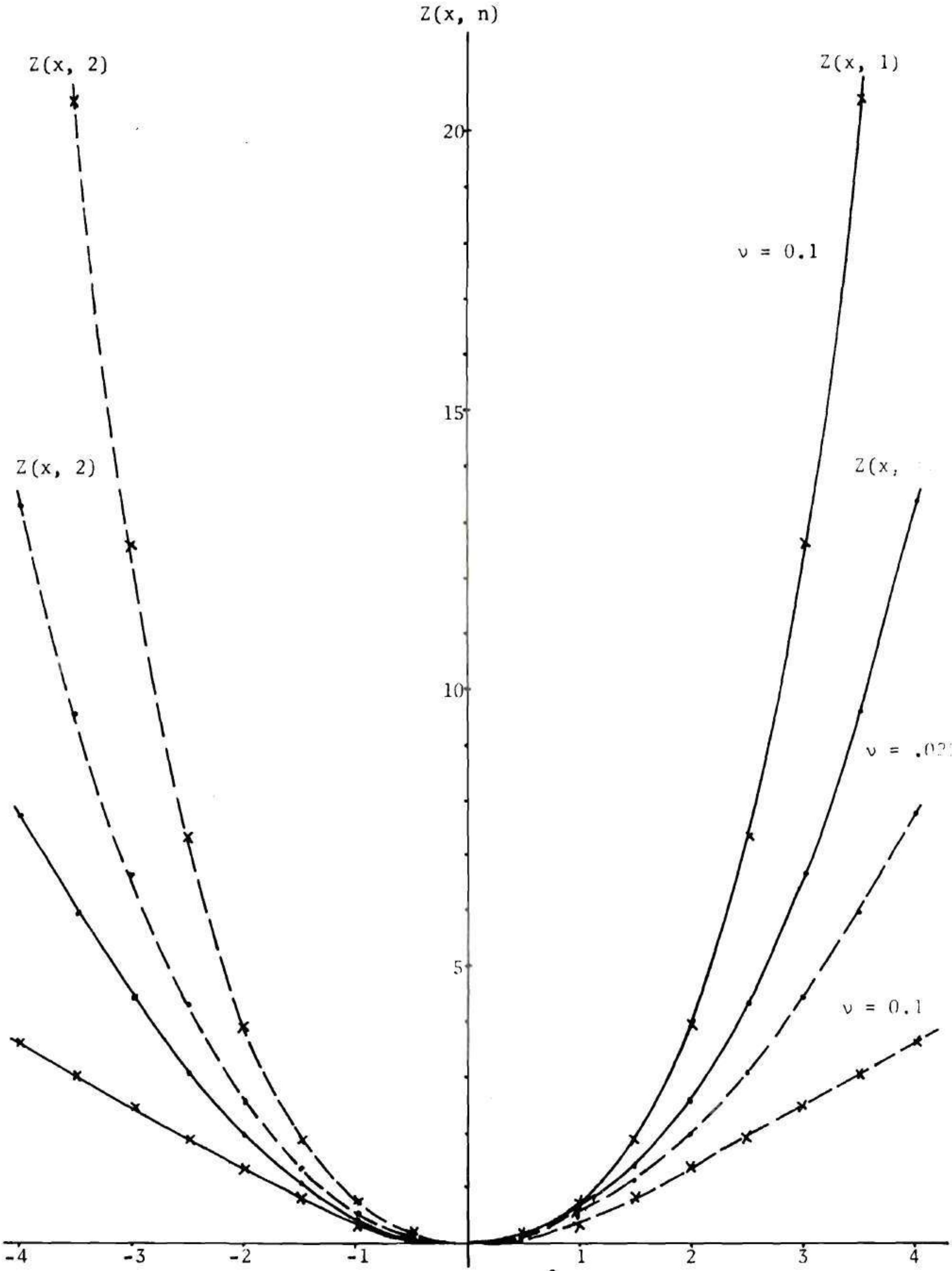


Figure A-16: Z(x, n) curves for $R(x) = x + \varepsilon x^2$, $f(x) = 1$, $(\nu = c_2/2m)$.

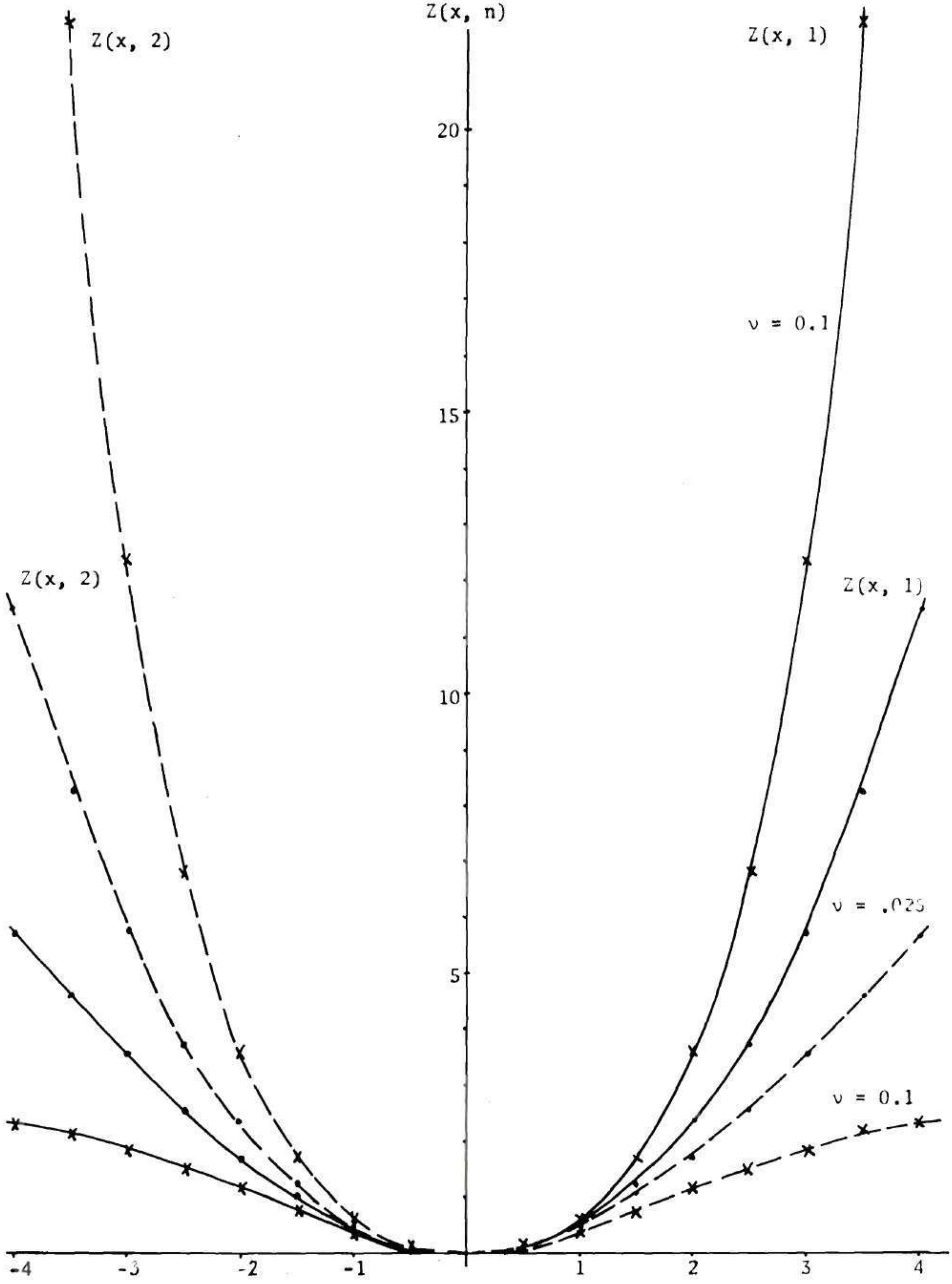


Figure A-17: $Z(x, n)$ curves for $R(x) = x$, $f(x) = 1 + \alpha \cos 4x$, $(\nu = c_2/2m)$.

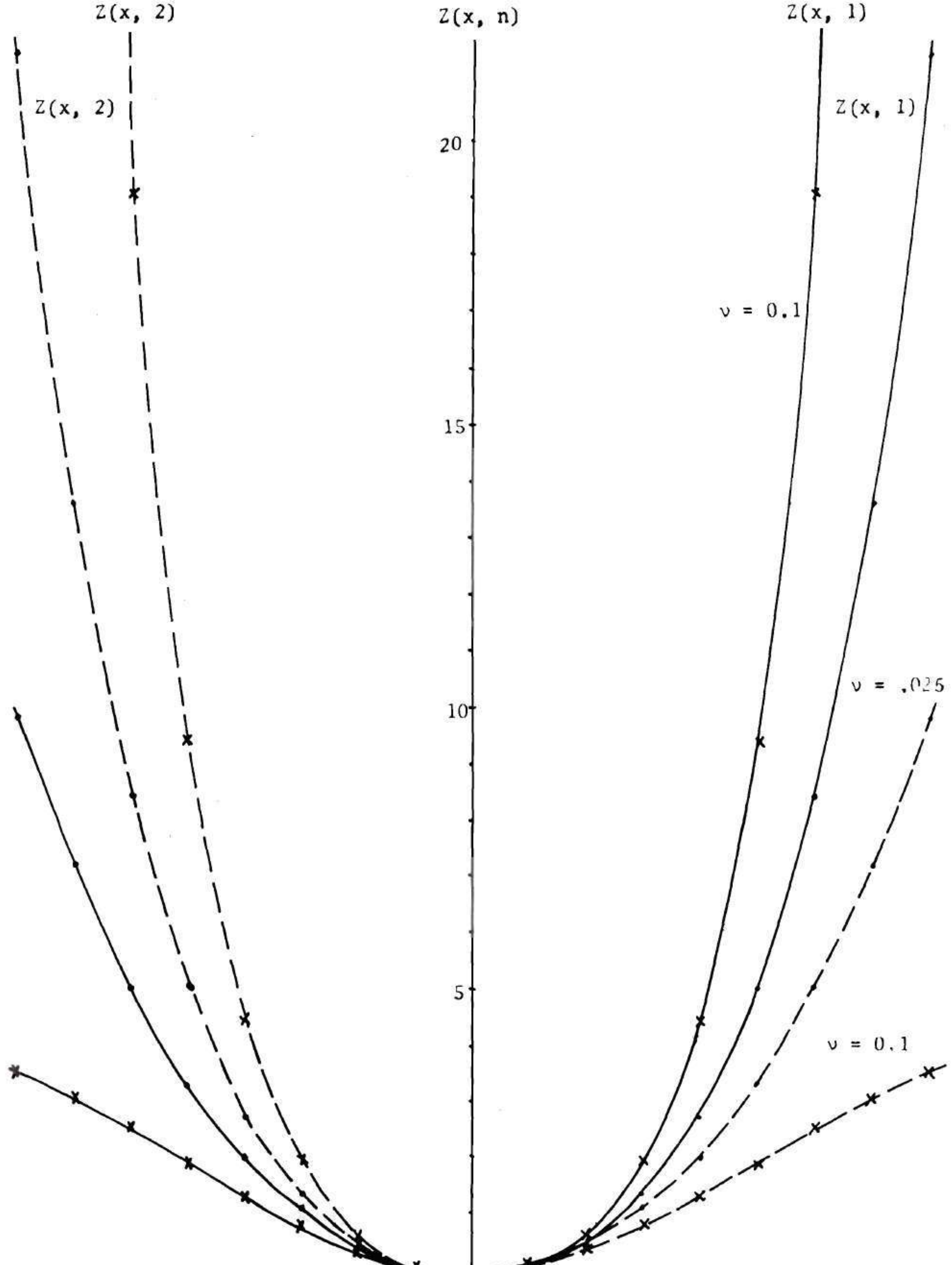


Figure A-18: Z(x, n) curves for $R(x) = x + \varepsilon x^3$, $f(x) = 1 + \alpha \cos 4x$, ($\nu = c_2/2m$).

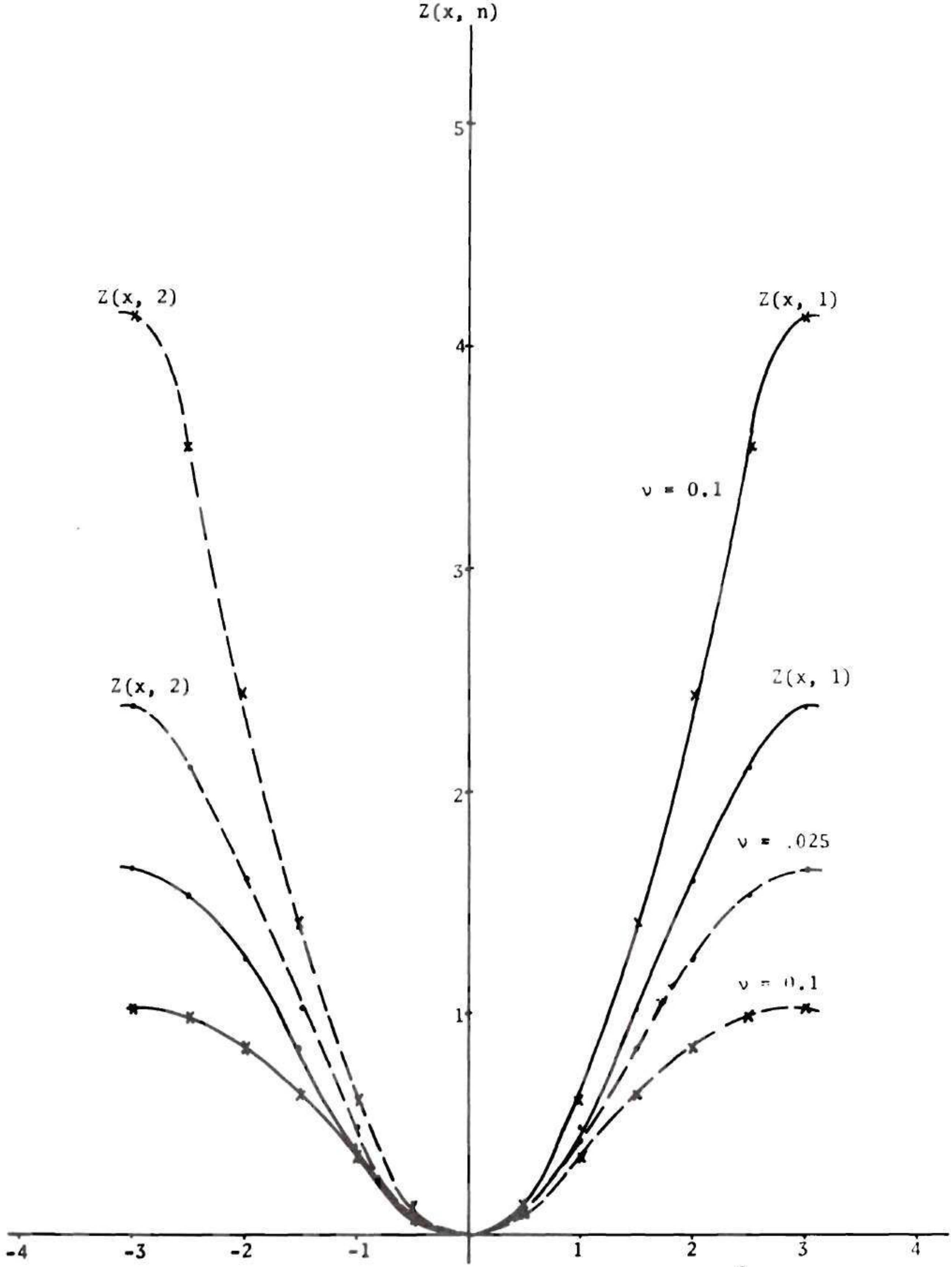


Figure A-19: $Z(x, n)$ curves for $R(x) = \sin(x)$, $f(x) = 1 + \alpha x^2$, $(\nu = c_2/2m)$.

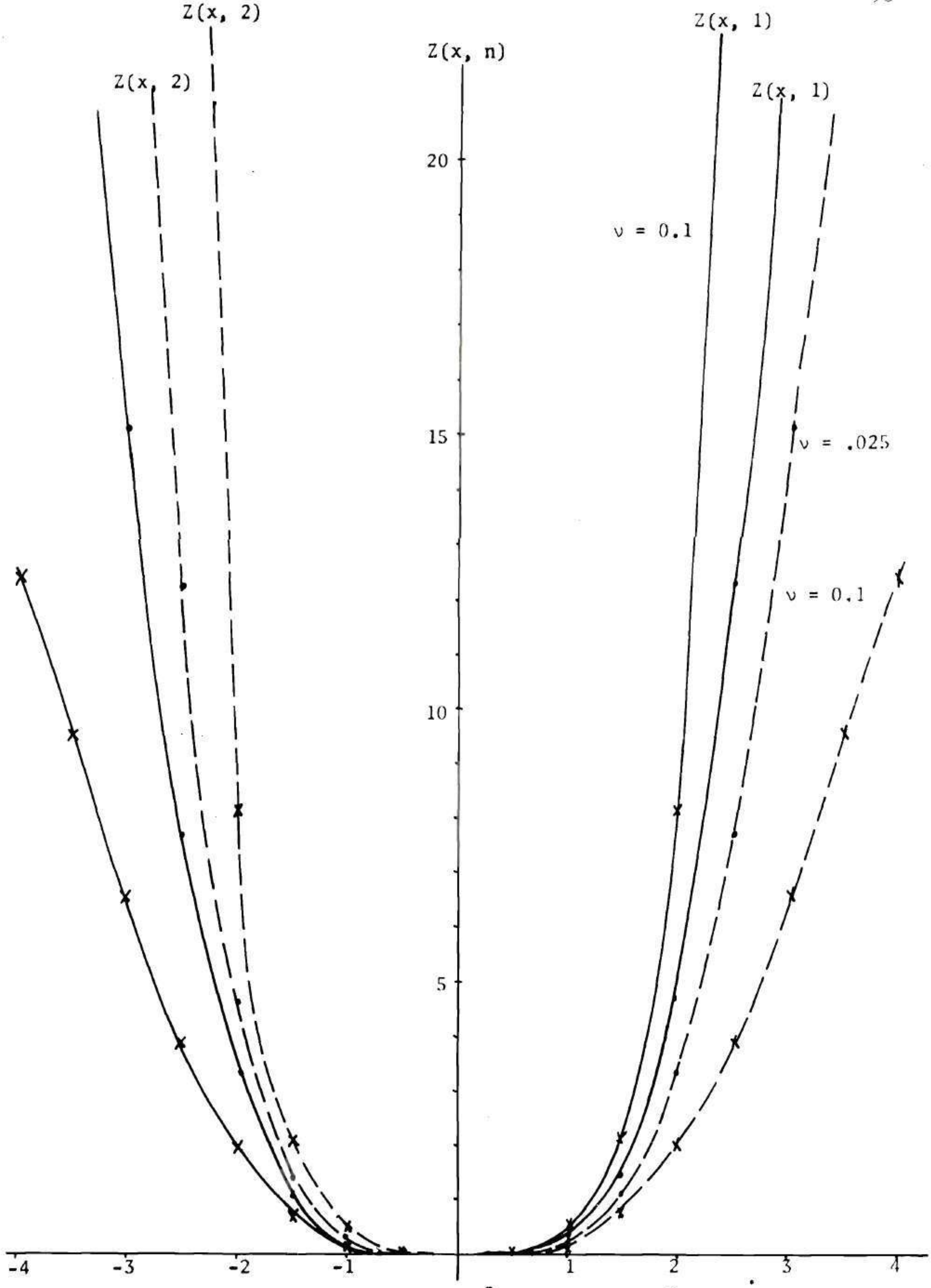


Figure A-20: $Z(x, n)$ curves for $R(x) = x^3$, $f(x) = 1 + \alpha x^2$, $(\nu = c_2/2m)$.

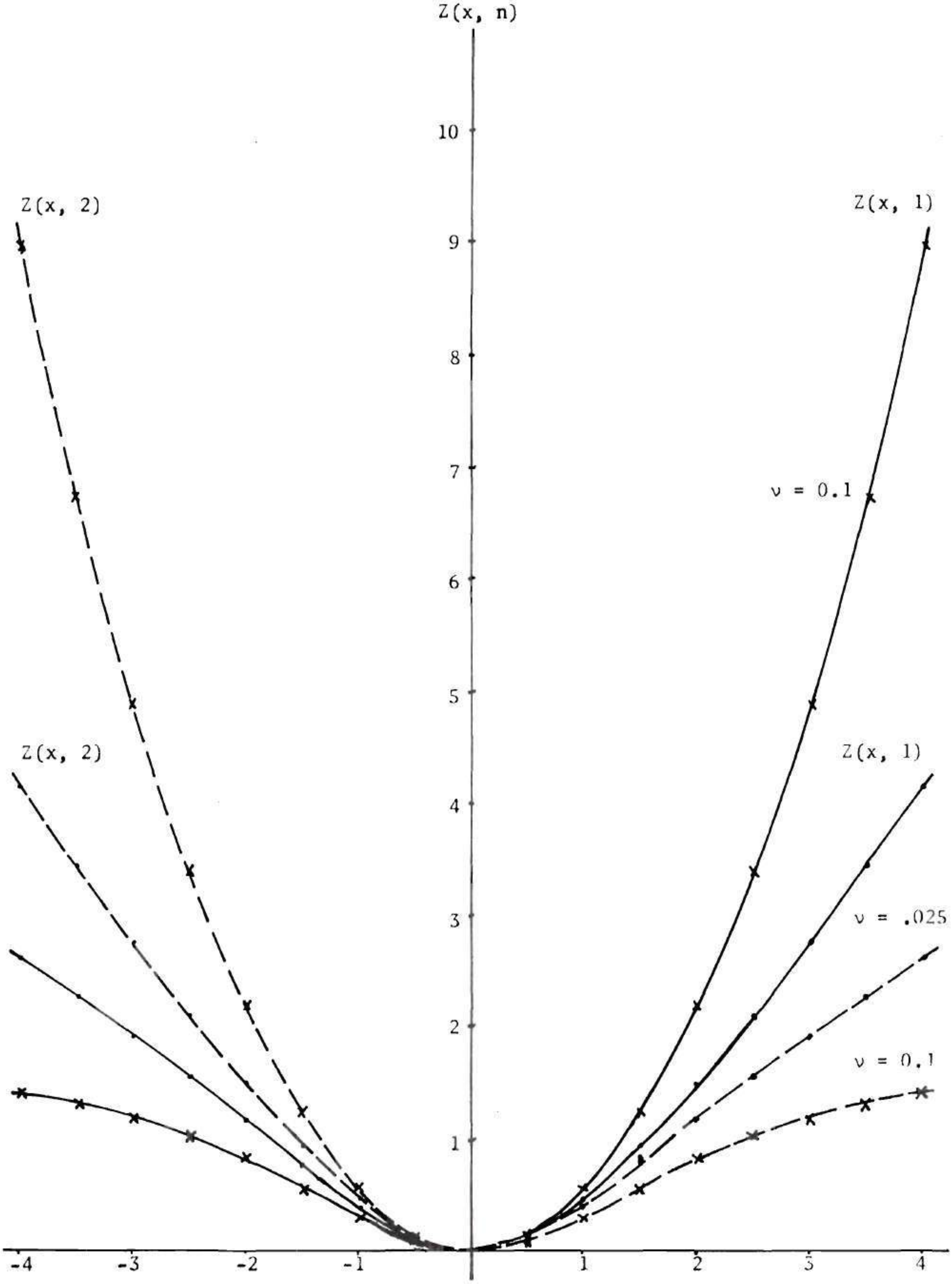


Figure A-21: Z(x, n) curves for $R(x) = \tanh x$, $f(x) = 1 + \alpha \cos 4x$, $(\nu = c_2/2m)$.

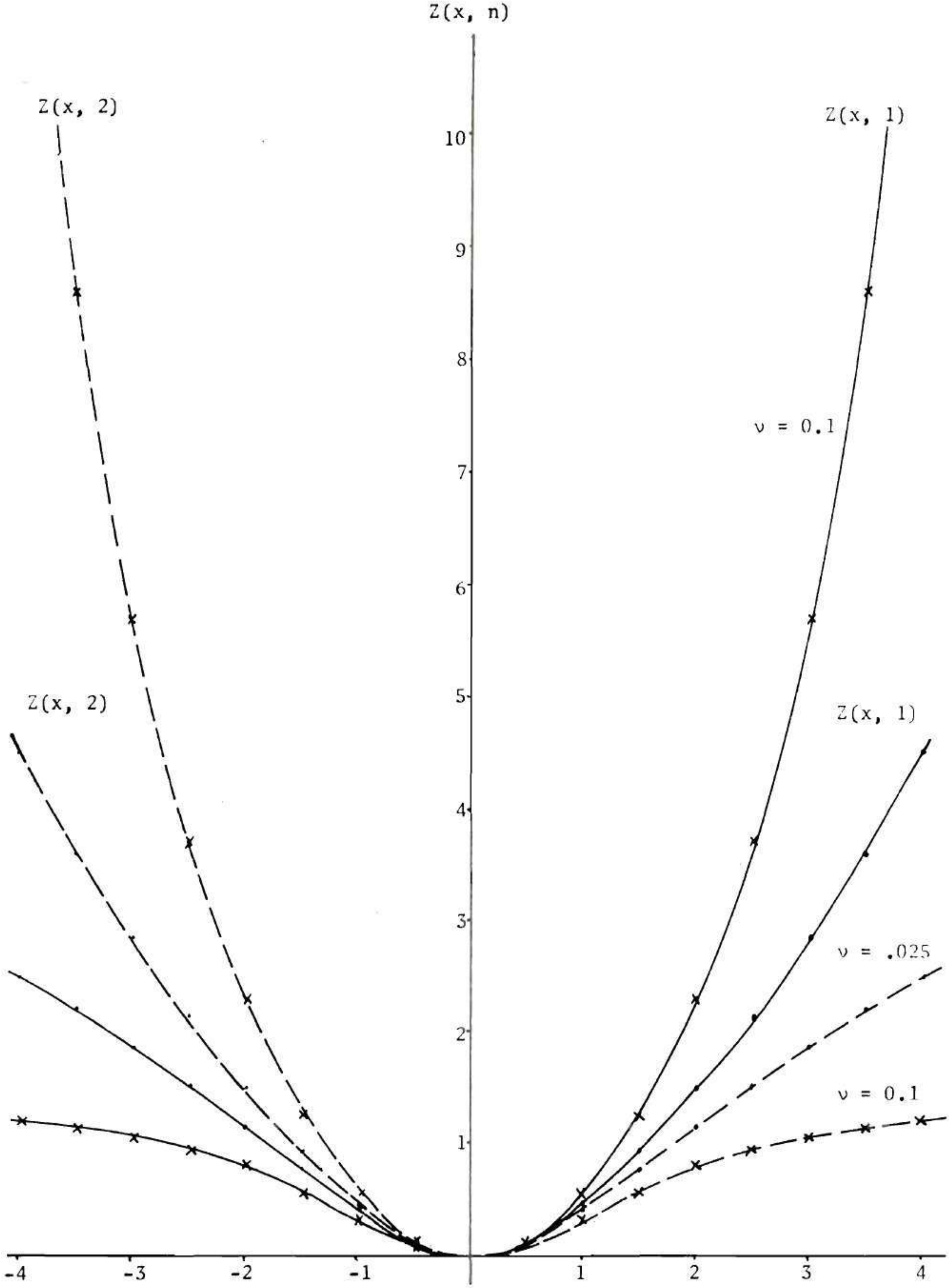


Figure A-22: $Z(x, n)$ curves for $R(x) = \tanh x$, $f(x) = 1 + \alpha x^2$, $(\nu = c_2/2m)$.